STABILITY

OF

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by

Mitu hiko Araki

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MITUHIKO ARAKI

Department of Electronics
Kyoto University
Kyoto, Japan

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PREFACE

In this thesis, a stability theory of composite systems is developed. The theory gives a powerful means to investigate the behavior of complex nonlinear control systems.

The word "stability" originates in mechanics where it characterizes the equilibrium of a rigid body. The equilibrium is called "stable" if the body returns to its original position after having been "disturbed", i.e. moved slightly from its position of rest. If the body tends toward a new position after a slight displacement, the equilibrium is called "unstable". The above concept of stability was generalized by Lyapunov for a motion of a general dynamical system. A motion of a dynamical system is a curve or "trajectory" C in some space. If any trajectory D starting near C remains near C, the trajectory C is stable. In addition, if D tends to C, the trajectory C is asymptotically stable. On the contrary, if D departs from C, the trajectory C is unstable. The stability treated in this thesis is the asymptotic stability in the sense of Lyapunov described above.

To investigate the stability of systems, or to be accurate, of the motion of systems has always been one of the most important themes in the theory of control systems. The importance of the stability problem in the control engineering results from the fact that we cannot strike out disturbances from real control systems. At the first stage of development, the theory of control systems was constituted by the classical linear theory, i.e. the theory of linear, time-invariant systems with a single input and a single output. The stability criterions named after Leonhard, Routh & Hurwitz, and Nyquist were, and have been, used to investigate such linear systems. The classical linear theory achieved great success and, still now, it constitutes the main part of the control theory in practical sense.

A demand for the nonlinear theory in the control engineering was first caused by undesirable nonlinear phenomenons, such as sustained oscillations, observed in various practical control systems. Later, the development of the optimal and adaptive control theory and of digital machines provoked a rapid increase of the intentional usage of nonlinear elements in control systems. Thereby, the demand for the nonlinear control theory became ever stronger. The first successful trial in the nonlinear control theory may be the describing function method which was mainly proposed and used in U.S.A. and Western Europe. On the other hand, the second method of Lyapunov was first used by Russian scientists and engineers for the investigation of nonlinear control systems. Especially, this method was applied successfully to the absolute stability problem, which is a typical nonlinear problem in the control engineering.

In the last ten years, a great number of researches on the stability problem of control systems were made on the basis of the linear and nonlinear theory described above. It would be impossible to survey all of them because the number is too great and the contents involved are too much widespread. However, we may classify them into three big streams. First, there is the endeavor to obtain ever sharper and more convenient stability criterions for nonlinear control systems of the classical type, i.e. nonlinear feedback systems with a single input and a single output. The famous theorem of Popov provoked a new development in this field. At present, various powerful stability criterions are available for such systems with a single nonlinear element. Second, there are studies on new types of stability different from or generalized from the stability in the sense of Lyapunov. The studies on the bounded-input-bounded-output stability, the $\mathcal{L}_2$-stability, the energy stability, the finite-time stability, the stability in the mean, the almost sure stability, the entropy stability and etc. are classified into this stream. Third, there is the endeavor to develop the theory so that more general class of systems can be treated. This kind of researches are often accompanied with the content which belongs to the second stream because, for the investigation of a new class of systems, a new concept is often required. The researches on multi-input-multi-output, linear, time-invariant systems, on distributed parameter systems and on stochastic control systems are classified into this stream. The research reported in this thesis also belongs to the third stream.

The recent development of science and industry is demanding and supplying more and more complex systems. From the standpoint of the theory of control systems, the complexity of systems can be understood to mean two things; first "largeness", i.e. the largeness of the dimension of the state vector, the number of inputs and outputs and the number of nonlinear and time-varying elements, and second "uncertainty", i.e. the uncertainty in observation and transmission, and the uncertainty of the system itself.

We may say that the urgent task imposed on the present control engineers is to overcome the difficulties caused by these two factors. This consideration is the main motivation of the researches on the stability problems which are classified into the third stream. For example, researches on multi-input-multi-output linear, time-invariant systems and on distributed parameter systems can be interpreted as trials to overcome the factor of "largeness", while the researches on stochastic systems are interpreted as trials to overcome the factor of "uncertainty".

The content of this thesis is also motivated by the above consideration. Here, we aim at establishing a stability theory which can give explicit results meaningful in practical sense for complex high-dimensional systems containing many nonlinear elements. For this purpose, we utilize the fact that many of complex engineering systems are made up of an interconnection of simple subsystems, in short, that they have the "composite structure". The theory developed here assumes that the system in question has the composite structure,and gives a systematic way to know the behavior of the system from the knowledge on the individual subsystems and the interconnection among them. By this composite systems theory, we will obtain various interesting results on the transient behavior of complex nonlinear systems, which were difficult to be analysed before.

CONTENTS

GLOSSARY

Symbols

a, b, --- (lightface, lower case Roman) : scalors

α, β, --- (lightface, lower case Greek) : scalors

$\mathbf{a}, \mathbf{b}$, --- (boldface, lower case Roman) : vectors

A, B, --- (lightface, upper case Roman) : matrixes

Exceptions

$N(\ ), M(\ )$: norm-like scalor-valued functions

K : a scalor constant

$\mathbf{x}$: the state vector of the composite system

$\mathbf{x}_i$: the state vector of the i-th subsystem

x_{ik} : the k-th component of the vector $\mathbf{x}_i$

t : the real-valued variable indicating the time

τ : the integer-valued variable indicating the sampling instant

n : the number of subsystems which compose the composite system

i, j : the integers which identify the subsystems (They take the values 1,---,n.)

Definitions

system CCS p.10

system SCS p.14

Chapter 1 Introduction

In this chapter, a review will be made upon the stability criterions available at present for multidimensional systems with many nonlinearities, and then the outline of the content will be described. In our review, only the main aspects of important criterions, which assure the asymptotic stability in the large (a.s.i.l.)* of systems containing many nonlinearities, will be pointed out. As for a more general list of recent researches in the field of deterministic stability problems and their detailed review, the readers are refered to the literatures by Brockett [1] and by Pyatnitskii [2].

Sec. 1.1. Stability Criterions for Continuous-Time Systems with Many Nonlinearities

The second method of Lyapunov using the "quadratic form plus integral of nonlinearity" type Lyapunov functions and the frequency domain method proposed by Popov

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* Henceforce, we abbreviate both "asymptotic stability in the large" and "asymptotically stable in the large" to "a.s.i.l." As for the definition of a.s.i.l., refer to [44] (for continuous-time systems) and [29] (for sampled-data systems).

have been two main tools of the stability investigation of control systems with a single nonlinearity. [3-6] Those two methods are generalized and used also for the investigation of control systems with many nonlinearities.

The application of the second method of Lyapunov to systems with many nonlinearities was reported by Letov [6], Sultanov [7], Tokumaru & Saito [8], and other authors (cf. [2]). The stability criterion obtained by this method reads as "the system is a.s.i.l. if there exist a set of constants as many as nonlinearities and a positive definite matrix of order equal to the dimension of the state vector such that a certain system of quadratic equations (generalized Lur'e resolving equations) has real solutions." This kind of criterion is theoretically interesting as a generalization of single nonlinearity case. However, from the practical viewpoint, it is not so useful because of the following reasons.

1) To know whether the system of quadratic equations has real solutions is a formidable work.
2) The criterion contains many arbitrary parameters, and we don't have any practical means for establishing their existence.
3) The increase of the dimension of the system causes a rapid increase of the required computation.

Besides the above way, the second method of Lyapunov can be applied for the investigation of systems with many nonlinearities in various ways (Krosovskii's theorem [9] or the variable gradient method [65-67]). However, its direct application suffers more or less from the same kind of difficulties described above.

The generalization of the frequency domain criterion for systems with many nonlinearities is reported by Popov (cf. [2]), Tokumaru & Saito [8], Jury & Lee [10], Anderson [11], Yakubovich [12] and Partovi & Nahi [13]. The obtained stability criterion reads, in its most general form, as "the system is a.s.i.l. if there exist a set of constants as many as the nonlinearities and another set of constants as many as the constraints on the nonlinearities such that a certain quadratic form depending upon the complex variable s is negative definite for all $s = i\omega$ $(-\infty < \omega < \infty)$". The relation between this criterion and the criterion obtained by the second method of Lyapunov was discussed by Sultanov [7], Tokumaru & Saito [8] and others (cf. [2]). This criterion seems suited for a detailed study of a comparatively low-dimensional systems with a few nonlinearities. In reality, a method of establishing the negative definiteness of the quadratic form for $s = i\omega$ was proposed by Lindgren & Pinkos [14] and Hirai & Kurematsu

[15,16] for the case of two nonlinearities. However, the application of this criterion to high-dimensional systems containing several nonlinearities meets with great computational difficulties; i.e.

1) We don't have any practical means for establishing the negative definiteness of the quadratic form when the number of nonlinearities is greater than two.

2) The criterion contains many arbitrary parameters and we don't have any practical means for establishing their existence.

The main reason of the difficulties involved in the two methods described above is that we take too much detaile knowledge on the whole system into consideration and do not have any systematic way to utilize the available information Reflecting upon this fact, Bailey [17] and some other author [18,19] proposed a new method of investigating high-dimensional systems with many nonlinearities. Their method is an attempt to utilize the fact that many of complex engineer ing systems are made up of an interconnection of simple subsystems. It can be named the "decomposition" method of stability investigation.

Bailey [17] reduced the problem of a high-dimensional system with many nonlinearities to a problem of a linear system by decomposing the original system into several lower-dimensional subsystems with fewer nonlinearities and assuming the knowledge of the Lyapunov functions associated with individual subsystems. His main tool was the vector Lyapunov function method [20,21]. His criterion reads as "the system is a.s.i.l. if a certain auxiliary linear system of order equal to the number of subsystems is a.s.i.l." Thus, his method presupposes the previous researches which prepare various methods to construct a Lyapunov function for a comparatively low-dimensional system with a few nonlinearities, and contains the step of investigating the stability of linear, time-invariant system described by a matrix-vector type equation, which is generally a formidable work (cf. p. 244 of [22],[23]). Bailey's method was applied to several practical examples by Piontkovskii & Rutkovskaya [18]. On the other hand, Michel reported a generalization of the second method of Lyapunov for the investigation of the stability, transient behavior and trajectory bounds of composite systems [19]. Michel's theorems are important from the theoretical point of view, but rather remote from practical application.

We can say that the researches reported hitherto using the "decomposition" method have not attained the level such that they can present practically powerful criterions, but are indicating a promising direction for the investigation of high-dimensional systems with many nonlinearities because of the following reasons:

1) By decomposing the whole system into subsystems, the method enables us to deal with comparatively high-dimensional systems compared with the two classical methods described at the beginning of this section.

2) The detailed study on a subsystem can be utilized in the investigation of the whole system. Even if some part of the system is changed, the previous results on the other parts of the system can be used almost unchanged for the investigation of the new system.

Sec. 1.2. Stability Criterions for Sampled-Data Systems with Many Nonlinearities

The stability theory for sampled-data control systems has been developed almost in parallel with the one of continuous-time systems, but still it contains a little different feature.

As for the frequency domain criterion, researches parallel with the continuous-time case were reported by Jury & Lee [10] and Yakubovich [24,25]. The obtained criterion contains same drawbacks as the one of the continuous-time case.

The application of the second method of Lyapunov to the sampled-data control systems with many nonlinearities was proposed and studied by Kalman & Bertram (Part II of [9]) Bubnicki [26,27] and Rozenvasser [28]. Their researches have a little different features from those of the continuous-time case. Kalman & Bertram and Bubnicki used the contraction concept and gave their criterions on the basis of totally linearized (cf. [29]) form of the difference equations. Rozenvasser also gave his criterion on the basis of the same linearized form of equations but made use of the concept of a majorant matrix. Rozenvasser also showed that his criterion and the criterion obtained by the contraction method are equivalent if the investigation can be made with respect to arbitrary large repetition periods. The above two methods are interesting and may be useful for a close examination of low-dimensional systems. But they generally have the following drawbacks.

1) If we want to obtain a less conservative condition, we must calculate the transition of the state over a larger period.

2) The increase of the dimension of the system causes a rapid increase of the required computation.

Between the above two methods, the Rozenvasser's majorant matrix method seems more useful for practical application because the contraction method has the drawback;

3) The criterion contains arbitrary parameters which

determines the norm and we don't have any practical means to establish the existence of the parameters satisfying the contraction condition.

The application of the "decomposition" method to sampled-data systems was mentioned by Michel and Wu [29] for a special type of sampled-data system, but any general investigation, which can give a concrete criterion, has not been reported except those by the author. [30-34]

Sec. 1.3. Outline of the Content

A new sort of stability theory suited for the investigation of multidimensional systems with many nonlinearities is developed in the following. The theory presented here was first established for sampled-data systems [30-34] and then it was extended and refined for continuous-time systems [35-37]. It is a same kind of theory as proposed by Bailey and others [17-19], but has attained a much advanced stage by the application of the theory of M-matrixes.

In Chapter 2, the concept of a composite system, on which the following theory is constructed, is introduced, and some remarks on the construction of the composite system model for engineering systems are made. In Chapter 3, a theorem, which gives a stability criterion for continuous-time composite systems, is given. The theorem is stated in Sec. 3.1., and proved in Sec. 3.2. The relation of our theorem to Bailey's result is discussed in Sec. 3.3. Generalization of the theorem, an estimate of the transient behavior and some examples are given in the following sections. In Chapter 4, a theorem, which gives a stability criterion for sampled-data composite systems, is given. The theorem is stated in Sec. 4.1. and proved in Sec. 4.2. Generalization of the theorem, an estimate of the transient behavior and some examples are given in the following sections. In Chapter 5, a system composed of subsystems, each of which contains single nonlinearity, is investigated. Especially, a continuous-time system composed of first and second order subsystems are studied in detail and a table, which gives constants necessary for the investigation of the stability of such systems, is calculated out. In Chapter 6, several concluding remarks are made. In Appendix, some theorems and definitions concerning M-matrixes are listed.

Chapter 2 Composite Control Systems

In this chapter, we give the equations of composite systems, upon which our theory is constructed.

Sec. 2.1. Continuous-Time Composite Systems

Let us consider the system CCS given by a set of n vector differential equations*

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} f_{ij}(x_j, t) \qquad i = 1, \cdots, n \tag{2-1}$$

Here x_i is a m_i-vector, t is a scalor belonging to $(-\infty, \infty)$, and $f_{ij}(x_j, t)$ is a vector function satisfying

$$f_{ij}(0, t) = 0 \tag{2-2}$$

We also assume the necessary smoothness requirements on $f_{ij}(x_j, t)$ such that the existence, uniqueness and continuity of the solutions of CCS are assured.

We can interpret that CCS is composed of n subsystems, each of which is described by the i-th equation of (2-1). In this sense, we say the system CCS is a

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* Throughout this thesis, we assume that the subscripts i and j take values $1, \cdots, n$ unless otherwise specified.

continuous-time composite system. Here, $f_{ii}(x_i, t)$ is interpreted as the dynamical property inherent to the i-th subsystem and u_i is interpreted as the input to the i-th subsystem where

$$u_i = \sum_{j;j \neq i} f_{ij}(x_j, t)$$

We can visualize the concept of a composite system as shown in Fig. 2.1.

For the convenience of the following disoussions, let us consider the <u>isolated</u> <u>i-th</u> <u>subsystem</u> of CCS, which is given by

$$\frac{dx_i}{dt} = f_{ii}(x_i, t) \tag{2-3-i}$$

By (2-2), $x_i = 0$ is the equilibrium of the isolated i-th

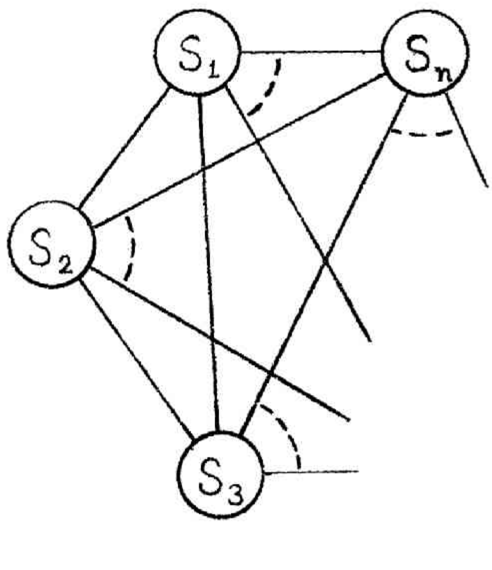

Fig. 2.1. A Composite System (S_i is the i-th subsystem)

subsystem. Put $x = x_1 \oplus \cdots \oplus x_n$ where $\oplus$ means the direct sum of vectors. Then, x is a state vector of CCS and $x = 0$ is an equilibrium by (2-2). Let us say the system CCS is a.s.i.l. if its trivial solution $x = 0$ is a.s.i.l. In the following, as a continuous-time composite system, we mainly treat the system CCS given by (2-1).

Now let us see how the equation (2-1) is obtained as a model of an engineering system. First, it is possible to argue as follows: "Construct a differential equation of the form

$$\frac{dx}{dt} = f(x, t) \quad , \quad f(0, t) = 0 \qquad (2\text{-}4)$$

for an engineering system. Then, find out a partitioning of the components of x such that (2-4) can be expressed in the form of (2-1)." A composite system model obtained in this way may be called a mathematical composite system model. In this case, each subsystem usually does not have any physical meaning.

Second, we can derive (2-1) in a more natural way. In many engineering problems, a complex system is made up of an interconnection of many transfer systems. Here, a transfer system S_i means an input-output device whose terminal variables are characterized by

$$\frac{dx_i}{dt} = h_{i1}(x_i, t) + h_{i2}(u_i, t) \tag{2-5}$$

$$y_i = h_{i3}(x_i, t) \tag{2-6}$$

where x_i, u_i and y_i are the state vector, the input vector and the output vector of the transfer system S_i. Now, suppose that the system in question is composed of n transfer systems S_i $(i=1,---,n)$ connected by the relation

$$u_i = h_{i4}(y_1, ---, y_n, v(t), t) \tag{2-7}$$

where $v(t)$ is an outer-input vector. Here, $v(t)$ is understood as a reference vector of the whole system and may be fixed. Then, by substituting (2-7) into (2-5), we obtain

$$\frac{dx_i}{dt} = h_{i1}(x_i, t) + h_{i5}(y_1, ---, y_n, t) \tag{2-8}$$

Here, if we can assume that h_{i5} is expressed as a sum of terms, each of which depends only upon the output of one transfer system and the time t, we obtain the equations of the form (2-1). Then, each transfer system can be understood as the subsystem of CCS in the sense defined at p.10. The last assumption will be refered to as the <u>superposition assumption of inputs to each subsystem</u>. A composite system model obtained in this way may be called a <u>physical composite system model</u>.

Usually, it is more profitable to use a physical composite system model than a mathematical composite system model. For, in the case of a physical composite system model, each subsystem stands for a component or physically meaningful part of the whole system, and the investigation of individual subsystems has an independent meaning. However, we cannot deny the possibility that a complex system is successfully analyzed by use of a mathematical composite system model while a physical composite system model can not give a satisfactory result.

As an example of application of the composite system model to an engineering system, we can refer to the analysis of the voltage and reactive power control of electric power systems [33]. We can find another example in the pitch and roll control problem of an air or undersea craft [14].

Sec. 2.2. Sampled-Data Composite Systems

Let us consider the system SCS given by a set of n vector difference equations

$$x_i(\tau+1)=\sum_{j=1}^{n} g_{ij}(x_j(\tau),\tau) \quad i=1,\cdots,n \tag{2-9}$$

Here x_i is a m_i-vector, τ is an integer which means the sampling instant and $g_{ij}(x_j, t)$ is a vector function satisfying

$$g_{ij}(0, \tau) = 0 \tag{2-10}$$

Like the continuous-time case, the system SCS can be interpreted as a sampled-data composite system. Here, the <u>isolated</u> i-th subsystem of SCS is given by

$$x_i(\tau+1) = g_{ii}(x_i, \tau) \tag{2-11-i}$$

By (2-10), $x_i = 0$ is the equilibrium of the isolated i-th subsystem. The vector $x = x_1 \oplus \cdots \oplus x_n$ is a state vector of SCS and $x = 0$ is an equilibrium of SCS. Here, we also say the <u>system SCS is a.s.i.l.</u> if its trivial solution $x = 0$ is a.s.i.l. In the following, as a sampled-data composite system, we mainly treat the system SCS.

For this sampled-data composite system model, we must note the following two points. First, when we derive a physical sampled-data composite system model, we must start, in most cases, from the continuous-time transfer system model as given by (2-5), (2-6) and (2-7). This would be rather evident if we note that the sampling devices are used only for the controling scheme. Second, the equation (2-9) presupposes that all the sampling devices of subsystems operate synchronously. This assumption may be satisfied when the subsystems are located geographically in a small region. But, in the case of composite system such as the electric power system, in which the subsystems are located over a large area, this assumption is rather unrealistic.

Chapter 3 Asymptotic Stability in the Large of Continuous-Time Composite Systems

Sec. 3.1. Main Theorem

Concerning a.s.i.l. of the continuous-time composite system CCS given by (2-1), we have the next theorem.

Theorem 3.1.

Consider the system CCS given by (2-1). Assume

(3-i) that, for each isolated subsystem (2-3-i), there exists a positive definite function $v_i(x_i, t)$ with continuous partial derivatives such that*

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* $\|x_i\|$ is the Euclidean norm of x_i and $\nabla_i v_i$ is the gradient of v_i with respect to x_i. Let x_{ik} the k-th component of x_i and let m stand for m_i. Then

$$\|x_i\| = \left\{ \sum_{k=1}^{m} x_{ik}^2 \right\}^{\frac{1}{2}}$$

$$\nabla_i v_i = \left(\frac{\partial v_i}{\partial x_{i1}}, \cdots, \frac{\partial v_i}{\partial x_{im}} \right)^t$$

$(dv_i/dt)_{(2-3-i)}$ means the derivative along the trajectory of (2-3-i).

$$\alpha_i(\|x_i\|) \leqq v_i(x_i, t) \leqq \beta_i(\|x_i\|) \qquad (3\text{-}1)$$

$$\left(\frac{dv_i}{dt}\right)_{(2-3-i)} \leqq -\gamma_i \|x_i\|^2 \qquad (3\text{-}2)$$

$$\|\nabla_i v_i\| \leqq \delta_i \|x_i\| \qquad (3\text{-}3)$$

where $\alpha_i(x)$ and $\beta_i(x)$ are continuous non-decreasing function of a positive argument x satisfying

$$\alpha_i(0) = \beta_i(0) = 0$$

$$\alpha_i(x) > 0, \ \beta_i(x) > 0 \quad \text{for} \quad x > 0$$

$$\alpha_i(x) \to \infty \quad \text{for} \quad x \to \infty$$

and γ_i and δ_i are positive constants,

(3-ii) and that there are non-negative constants ε_{ij} $(i \neq j)$ such that

$$\|f_{ij}(x_j, t)\| \leqq \varepsilon_{ij} \|x_j\| \quad i \neq j \qquad (3\text{-}4)$$

The system CCS is a.s.i.l. if

(3-iii) the matrix $A = (a_{ij})$ is an M-matrix, i.e. the leading principal minor determinants of A are all positive (cf. Definition A.1. in Appendix). Here, a_{ij}'s are given by

$$a_{ii} = \frac{\gamma_i}{\delta_i}, \quad a_{ij} = -\varepsilon_{ij} \qquad (3\text{-}5)$$

The theorem is proved in the next section. The assumption (3-i) means that each isolated subsystem is a.s.i.l. uniformly on the initial time, and the resulting a.s.i.l. of CCS is also uniform on the initial time [9]. The function $v_i(x_i, t)$ can be obtained by the results of previous researches on stability problems, especially by those on the absolute stability problem [3-5, 38-41]. The constant ε_{ij} of the assumption (3-ii) is the upper bound of the d.c. gain of the connection between subsystems. If the connection is linear and time-invariant, i.e. if $f_{ij}(x_j, t) = C_{ij}\, x_j \ (i \neq j)$ we obtain $\varepsilon_{ij} = \| C_{ij} \|$.*

In the stability condition (3-iii), a_{ii} is a kind of index of stability of each subsystem (cf. Sec. 3.5. or Chapter 5) and $|a_{ij}|\ (i \neq j)$ gives the strength of the connection. The assertion that A is an M-matrix means that diagonal elements are dominant in A [42,43]. So we can interpret Theorem 3.1. as asserting that, if the "stability" of subsystems is superior to the interconnection among subsystems, the composite system is stable. This interpretation well agrees with our intuition.

Now, let us see that the stability condition of Theorem 3.1. is a necessary and sufficient condition for a.s.i.l. of a certain linear system. Consider the simplest linear composite system with positive connections given by

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* $\| C_{ij} \|$ is the norm of a matrix defined by $\max\limits_{\|x_j\|=1} \| C_{ij}\, x_j \|$

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \ell_{ij} x_j \qquad i = 1, ---, n \tag{3-6}$$

Here, x_i is a scalor and ℓ_{ij}'s are constants satisfying

$$\ell_{ii} < 0 \ , \quad \ell_{ij} \geq 0 \quad (i \neq j) \tag{3-7}$$

This system is a special case of CCS, in which the state vector of each subsystem is one-dimensional. The isolated i-th subsystem is described by

$$\frac{dx_i}{dt} = \ell_{ii} \ x_i \tag{3-8-i}$$

Here, by putting $v_i = x_i^2$, the assumptions (3-i) and (3-ii) are satisfied and the constants are given by

$$\gamma_i = -2 \ \ell_{ii} \ , \ \delta_i = 2 \ , \ \varepsilon_{ij} = \ell_{ij} \tag{3-9}$$

Therefore, we obtain

$$a_{ij} = -\ell_{ij} \tag{3-10}$$

Then, it is evident, by the condition (A-vii) of Theorem A.1. in Appendix, that the condition (3-iii) of Theorem 3.1. is a necessary and sufficient condition for a.s.i.l. of the linear system (3-6).

Sec. 3.2. Proof of the Main Theorem and a Supplementary Theorem

First, we state and prove some important properties of matrixes with non-positive off-diagonal elements. Then, we prove Theorem 3.1.

3.2.1. M-Matrixes and Positive Definite Matrixes

An n-th order matrix $A=(a_{ij})$ with non-positive off-diagonal elements is called an M-matrix if and only if its leading principal minor determinants are all positive (Definition A.1. in Appendix). On the other hand, a necessary and sufficient set of conditions that an n-th order symmetric matrix $B=(b_{ij})$ be positive definite is also that its leading principal minor determinants are all positive. Therefore, if an M-matrix A is symmetric, A is positive definite and if the off-diagonal elements of a positive definite matrix B are all non-positive, B is an M-matrix. In addition, we have the next relation.

<u>Lemma</u> 3.1.

Let $A=(a_{ij})$ an n-th order matrix with non-positive off-diagonal elements, i.e.

$$a_{ij} \leqq 0 \qquad i \neq j \tag{3-11}$$

Let $B=(b_{ij})$ a symmetric matrix given by

$$B=\frac{1}{2}\left(A+A^{t}\right) \tag{3-12}$$

If B is positive definite, A is an M-matrix.

<u>Proof</u> : The lemma is evident for $n=1$. So, let us assume that the theorem is valid for $n<m$, and show that an m-th order matrix $A=(a_{ij})$ is an M-matrix if (3-11) is satisfied

and B given by (3-12) is positive definite. Then, we can conclude the lemma by the mathematical induction.

By the positive definiteness of B, the (m-1)-th order matrix at the left upper corner of B is also positive definite. Hence, the corresponding (m-1)-th order matrix at the left upper corner of A is an M-matrix by the assumption. Hence, the first $n-1$ of the leading principal minor determinants of A are all positive. Hence, if we can show $|A|>0$, we can conclude A is an M-matrix and the proof is complete.

By the ordinary reduction of the determinant, we obtain

$$|A| = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1m} \\ 0 & \\ \vdots & A^0 \\ 0 & \end{vmatrix} = a_{11}|A^0| \quad (3\text{-}13)$$

where $A^0=(a^0_{ij})$ is an (m-1)-th order matrix given by

$$a^0_{i-1\,j-1} = a_{ij} - \frac{a_{i1}\,a_{1j}}{a_{11}} \qquad i,j=2,\cdots,m \quad (3\text{-}14)$$

By the positive definiteness of B, we have

$$a_{11} = b_{11} > 0$$

Therefore, by (3-11),

$$a^0_{ij} \leqq 0 \qquad i,j=1,\cdots,m-1\ ;\ i\neq j$$

Here, let $B^0=(b^0_{ij})$ a symmetric matrix given by

$$B^0 = \frac{1}{2}\,(A^0 + A^{0t})$$

Then, by (3-14),

$$b^0_{i-1\,j-1} = \frac{1}{2}\left(a_{ij}+a_{ji}\right) - \frac{a_{i1}\,a_{1j} + a_{j1}\,a_{1i}}{2\,a_{11}} \qquad (3\text{-}15)$$

$$i, j = 2, ---, m$$

Here put

$$a_{1j} = b_{1j} + p_j \qquad j = 2, ---, m \qquad (3\text{-}16)$$

Then, by (3-12)

$$a_{i1} = b_{i1} - p_i \qquad i = 2, ---, m \qquad (3\text{-}17)$$

By substituting (3-16) and (3-17) into (3-15), we obtain

$$b^0_{ij} = b'_{ij} + \frac{1}{b_{11}}\, q_{ij} \qquad i, j = 1, ---, m-1 \qquad (3\text{-}18)$$

where

$$b'_{i-1\,j-1} = b_{ij} - \frac{b_{1i}\,b_{j1}}{b_{11}}$$

$$i, j = 2, ---, m$$

$$q_{i-1\,j-1} = p_i\, p_j$$

Since B is positive definite, the (m-1)-th order matrix $B'=(b'_{ij})$ is positive definite. (Note that a principal minor determinant of k-th order of B' corresponds to a principal minor determinant of (k+1)-th order of B.) the (m-1)-th order matrix $Q=(q_{ij})$ is also positive definite because Q is expressed as

$$Q = \mathbb{p}\cdot\mathbb{p}^t$$

where $\mathbb{p} = (p_2, ---, p_m)^t$. Hence, the matrix B^0 is positive definite by (3-18). Hence, the (m-1)-th order matrix A^0 is an M-matrix by the assumption. Hence, $|A^0| > 0$. Hence, $|A| > 0$ by (3-13). (Q.E.D.)

The direct inverse of Lemma 3.1. is not valid. That is, even if A is an M-matrix, $B = (A+A^t)/2$ is not always positive definite, as is exemplified by

$$A = \begin{pmatrix} 1 & -0.2 \\ -4 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & -2.1 \\ -2.1 & 1 \end{pmatrix}$$

But, we have the next lemma, which is a weakened inverse of Lemma 3.1.

Lemma 3.2.

Let $A = (a_{ij})$ an n-th order M-matrix. There exists a set of positive numbers $w_1, ---, w_n$ such that the matrix B given by

$$B = \frac{1}{2}\left\{ diag\,(w_1, ---, w_n)\cdot A + A^t\cdot diag\,(w_1, ---, w_n)\right\} \tag{3-19}$$

is positive definite. Here $\mathrm{diag}(w_1, ---, w_n)$ means a diagonal matrix with the diagonal elements $w_1, ---, w_n$.

Proof : By the condition (A-ii) and (A-iv) of Theorem A.1.

in Appendix, there are two sets of positive numbers $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ such that $c_1 > 0, \cdots, c_n > 0$ and $d_1 > 0, \cdots, d_n > 0$ where the vectors $\mathbb{x} = (x_1, \cdots, x_n)^t$ $\mathbb{y} = (y_1, \cdots, y_n)^t$, $\mathbb{c} = (c_1, \cdots, c_n)^t$ and $\mathbb{d} = (d_1, \cdots, d_n)^t$ are associated by the relations

$$\mathbb{c} = A\mathbb{x} \ , \ \mathbb{d} = A^t \mathbb{y}$$

Here put

$$w_i = \frac{y_i}{x_i} \tag{3-20}$$

Then, we obtain

$$\begin{aligned} B\mathbb{x} &= \frac{1}{2}\left\{ diag(w_1, \cdots, w_n) A\mathbb{x} + A^t diag(w_1, \cdots, w_n)\mathbb{x} \right. \\ &= \frac{1}{2}\left\{ diag(w_1, \cdots, w_n)\, \mathbb{c} + \mathbb{d} \right\} \end{aligned} \tag{3-21}$$

Hence, the components of $B\mathbb{x}$ are all positive. Hence, by the condition (A-ii) of Theorem A.1., the matrix B is an M-matrix. Since B is symmetric, B is positive definite. (Q.E.D.)

3.2.2. Proof of Theorem 3.1. and a Supplementary Theorem

As the first step, let us prove the next lemma.

<u>Lemma</u> 3.3.

Consider the system CCS given by (2-1). Assume (3-i) and (3-ii). The system CCS is a.s.i.l. if

(3-iv) there exists a set of n positive numbers $w_1, \cdots$

$\cdots, w_n$ such that the matrix $B = (b_{ij})$ is positive definite. Here, b_{ij} 's are given by

$$\left.\begin{aligned} b_{ii} &= w_i \gamma_i \\ b_{ij} &= -\frac{1}{2}\left(w_i \delta_i \varepsilon_{ij} + w_j \delta_j \varepsilon_{ji} \right) \quad i \neq j \end{aligned}\right\} \qquad (3\text{-}22)$$

<u>Proof</u> : Put

$$V(x, t) = \sum_i w_i v_i(x_i, t) \qquad (3\text{-}23)$$

By (3-1), there exist continuous non-decreasing functions $\alpha(x)$ and $\beta(x)$ satisfying

$$\alpha(0) = \beta(0) = 0$$

$$\alpha(x) > 0, \ \beta(x) > 0 \qquad \text{for} \quad x > 0$$

$$\alpha(x) \to \infty \qquad \text{for} \quad x \to \infty$$

such that

$$\alpha(\|x\|) \leqq V(x, t) \leqq \beta(\|x\|) \qquad (3\text{-}24)$$

Therefore, if we can show that $(dV/dt)_{(2\text{-}1)}$ is negative definite, we can conclude the system CCS is a.s.i.l. uniformly on the initial time by Lyapunov's theorem on stability. (cf. Theorem 1. of [9])

From (3-23) and (2-1), we obtain

$$\left(\frac{dV}{dt}\right)_{(2\text{-}1)} = \sum_i w_i \left\{ \left(\frac{dv_i}{dt}\right)_{(2\text{-}3\text{-}i)} + \sum_{j;j\neq i} (\nabla_i v_i)^t f_{ij}(x \right.$$

$$(3\text{-}25)$$

By (3-3) and (3-4)

$$\left|(\nabla_i v_i)^t f_{ij}(x_j, t)\right| \leqq \delta_i \varepsilon_{ij} \|x_i\| \|x_j\| \qquad (3\text{-}26)$$

By applying the inequalities (3-2) and (3-26) to (3-25), we obtain

$$\left(\frac{dV}{dt}\right)_{(2-1)} \leqq -\sum_{i,j} b_{ij} \|x_i\| \|x_j\| \qquad (3\text{-}27)$$

, which tells that $(dV/dt)_{(2-1)}$ is negative definite.

(Q.E.D.)

As the second step, let us prove the next lemma.

<u>Lemma</u> 3.4.

The condition (3-iii) is equivalent to the condition (3-iv).

<u>Proof</u> : First let us show that (3-iv) follows (3-iii). If (3-iii) is satisfied, by Lemma 3.2., there exists a set of positive numbers $w'_1, \cdots, w'_n$ such that the matrix $B = (b_{ij})$ is positive definite where

$$\begin{aligned} b_{ii} &= w'_i \frac{\gamma_i}{\delta_i} \\ b_{ij} &= -\frac{1}{2}\left(w'_i \varepsilon_{ij} + w'_j \varepsilon_{ji}\right) \end{aligned} \qquad (3\text{-}28)$$

Then, by putting

$$w_i = \frac{w'_i}{\delta_i} \qquad (3\text{-}29)$$

, we can see that (3-iv) is satisfied.

Next, let us show that (3-iii) follows (3-iv). If the matrix $B=(b_{ij})$ given by (3-22) is positive definite, the matrix $A'=(a'_{ij})$ is an M-matrix by Lemma 3.1. where

$$a'_{ii} = w_i \gamma_i \quad , \; a'_{ij} = - w_i \delta_i \varepsilon_{ij} \quad (i \neq j) \qquad (3\text{-}30)$$

Then, the matrix A given by (3-5) is an M-matrix by Theorem A.3. in Appendix because A is obtained from A' if we multiply each row of A' by $1/w_i \delta_i$. (Q.E.D.)

By Lemma 3.3. and 3.4., Theorem 3.1. is evident.

Here, let us re-examine the proof of Lemma 3.1. If we examine the reduction of (3-27) from the three relations (3-2), (3-3) and (3-4), we can see that the same reduction is possible even if $\|x_i\|$ is replaced with a positive definite function $M_i(x_i)$ of x_i . In addition, if the k-th components of $f_{ij}(x_j, t)$ $(j=1,\cdots,n\,;\, j \neq i)$ are all identically 0 , we can obtain (3-27) even if we ommit the k-th component of $\nabla_i v_i$ from the relation (3-3). Therefore, we obtain the next theorem.

<u>Theorem</u> 3.2.

For the functions $f_{ij}(x_j, t)$ $(j=1,\cdots,n\,;\, j \neq i)$, assume*

---

* $f_{ijk}(x_j, t)$ is the k-th component of $f_{ij}(x_j, t)$.

$$f_{ijk}(x_j, t) \equiv 0 \quad k = k_1^{(i)}, \cdots, k_{p_i}^{(i)} \; ; \; j = 1, \cdots, n \; ; \; j \neq$$

In Theorem 3.1., we can replace (3-2), (3-3) and (3-4) by

$$\left(\frac{dv_i}{dt}\right)_{(2-3-i)} \leq -\gamma_i \, M_i(x_i)^2 \qquad (3\text{-}2\text{-}g)$$

$$\| \text{modified}(\nabla_i v_i) \| \leq \delta_i \, M_i(x_i) \qquad (3\text{-}3\text{-}g)$$

$$\| \mathbf{f}_{ij}(x_j, t) \| \leq \varepsilon_{ij} \, M_j(x_j) \qquad (3\text{-}4\text{-}g)$$

Here, $M_i(x_i)$ is a positive definite continuous function of x_i and modified($\nabla_i v_i$) is a vector made from $\nabla_i v_i$ by replacing k-th components ($k = k_1^{(i)}, \cdots, k_{p_i}^{(i)}$) with 0.

Sec. 3.3. Comparison with the Vector Lyapunov Function Method

Here, let us compare the stability condition of Theorem 3.1. with the one obtained by Bailey [17] using the vector Lyapunov function method. Bailey gave his theorem only for the composite systems with linear, time-invariant connections, i.e. for such systems in which $\mathbf{f}_{ij}(x_j, t) = C_{ij} x_j \; (j \neq i)$. However, his theorem can be easily extended for the systems with general connections under the assumption (3-ii). [18] The extended Bailey's theorem reads as follows.

Theorem 3.3.a. (Bailey)*

Consider the system CCS given by (2-1). Assume (3-i) and that the functions $\alpha(\cdot)$ and $\beta(\cdot)$ are given as

$$\left.\begin{aligned} \alpha_i(\|x_i\|) &= \alpha_i \|x_i\|^2 \\ \beta_i(\|x_i\|) &= \beta_i \|x_i\|^2 \end{aligned}\right\} \tag{3-31}$$

where α_i and β_i on the righthand side are positive constants. Assume also (3-ii). The system CCS is a.s.i.l. if

(3-v) the auxiliary linear system given by

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} l'_{ij} x_j \qquad i = 1, ---, n \tag{3-32}$$

$$\left.\begin{aligned} l'_{ii} &= -\frac{\gamma_i}{2\beta_i} \\ l'_{ij} &= \begin{cases} \dfrac{\delta_i^2}{2\alpha_i \gamma_i} \displaystyle\sum_{k=1; k\neq i}^{n} \varepsilon_{ik}^2 & i\neq j, \varepsilon_{ij}\neq 0 \\ 0 & i\neq j, \varepsilon_{ij}=0 \end{cases} \end{aligned}\right\} \tag{3-33}$$

---

* This theorem is an extension of Theorem 6.3. of the literature [17]. The quantities l'_{ij} and ε_{ij} of the above theorem correspond respectively to a_{ij} and $\|c_{ij}\|$ of [17]. In the literature [17], the factor "2" in the denominator of the equation defining a_{ij} for $i\neq j$ is missing. From the equation just above eq. (6.10) of [17], it is evident that a_{ij} must be given by the equation like (3-33).

is a.s.i.l.

Here, a necessary and sufficient condition for (3-v) is, by the condition (A-vii) of Theorem A.1. in Appendix, that the matrix $(-\ell'_{ij})$ is an M-matrix. By Theorem A.3., this is equivalent to

(3-vi) the matrix $D=(d_{ij})$ is an M-matrix where

$$\left.\begin{aligned} d_{ii} &= \frac{\alpha_i}{\beta_i}\left(\frac{\gamma_i}{\delta_i}\right)^2 \\ d_{ij} &= \begin{cases} -\sum_{k=1;\,k\neq i}^{n} \varepsilon_{ik}^{2} & i\neq j,\ \varepsilon_{ij}\neq 0 \\ 0 & i\neq j,\ \varepsilon_{ij}=0 \end{cases} \end{aligned}\right\} \quad (3\text{-}34)$$

Therefore, we can convert the above theorem to the next form.

<u>Theorem</u> 3.3.b.

Under the assumptions of Theorem 3.2.a., the system CCS is a.s.i.l. if (3-vi).

Here, let us compare our Theorem 3.1. with Theorem 3.3. Theorem 3.3. uses more information than Theorem 3.1; i.e. Theorem 3.3. uses the information about the functions $\alpha_i(x)$ and $\beta_i(x)$ while Theorem 3.1. requires only the existence of such functions. This indicates the possibility that Theorem 3.3. gives less conservative stability condition than Theorem 3.1. when such information as given by (3-31) is available concerning the functions $\alpha_i(x)$ and $\beta_i(x)$. However, the contrary is the case in reality, i.e. Theorem 3.3. is generally more conservative than Theorem 3.1., as shown in the following.

The assertion that Theorem 3.3. is more conservative than Theorem 3.1. means that (3-iii) follows (3-vi) but (3-vi) does not always follow (3-iii). First, let us show that (3-iii) follows (3-vi). Let us suppose that D given by (3-34) is an M-matrix for $\alpha_i=\alpha_i^*$, $\beta_i=\beta^*_i$ $\gamma_i=\gamma^*_i$, $\delta_i=\delta^*_i$ and $\varepsilon_{ij}=\varepsilon^*_{ij}$, and show that A given by

$$a_{ii}=\frac{\gamma_i^*}{\delta^*_i} \quad , \quad a_{ij}=-\varepsilon^*_{ij} \quad (i\neq j) \qquad (3\text{-}35)$$

is also an M-matrix. Let $D'=(d_{ij})$ a matrix given by

$$\left.\begin{aligned} d'_{ii} &= \left(\frac{\gamma^*_i}{\delta^*_i}\right)^2 \geq d_{ii} & \\ d'_{ij} &= d_{ij} & i\neq j \end{aligned}\right\} \qquad (3\text{-}36)$$

Then, by Theorem A.2. in Appendix, D' is an M-matrix since we suppose D is an M-matrix. Now, consider the simplest linear composite system (3-6) and let

$$\ell_{ii}=-\frac{\gamma_i^*}{\delta_i^*} \quad , \quad \ell_{ij}=\varepsilon^*_{ij} \quad (i\neq j) \qquad (3\text{-}37)$$

Put

$$v_i = \frac{1}{2} \delta_i^* x_i^2$$

Then we obtain

$$\left(\frac{dv_i}{dt}\right)_{(3-8-i)} = -\gamma_i^* x_i^2 \ , \ \nabla_i v_i = \delta_i^* x_i$$

Therefore, all the assumptions of Theorem 3.3. are satisfied for this linear system where the constants are given by

$$\alpha_i = \beta_i = \frac{1}{2} \delta_i^* \ , \ \gamma_i = \gamma_i^* \ , \ \delta_i = \delta_i^*$$
$$\varepsilon_{ij} = \varepsilon_{ij}^*$$

Since the matrix D' given by (3-36) is an M-matrix, this linear system is a.s.i.l. by Theorem 3.3. Then, by the condition (A-vii) of Theorem A.1., the matrix $(-l_{ij})$ is an M-matrix. Here, by comparing (3-35) and (3-37), we obtain $A=(-l_{ij})$, hence A given by (3-35) is an M-matrix.

Now, let us show that (3-vi) does not always follow (3-iii). This is easily exemplified by the cases, in which the ratios α_i/β_i are very small. But, even when $\alpha_i=\beta_i$ for all i, we have cases in which (3-vi) does not follow (3-iii), as follows. First, note that, if $\alpha_i=\beta_i$ for all i, the matrix A and D are associated by the relation

$$d_{ii} = a_{ii}^2$$

$$d_{ij} = \begin{cases} -\sum_{k=1;k\neq i}^{n} a_{ik}^{2} & i\neq j, a_{ij}\neq 0 \\ 0 & i\neq j, a_{ij}=0 \end{cases}$$

Here, we can give the next example.

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -0.8 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -2 & -2 \\ -1.64 & 1 & -1.64 \\ 0 & 0 & 1 \end{pmatrix}$$

Sec. 3.4. Generalization of the Main Theorem

Two generalized forms of Theorem 3.1. are given in this section. First, the theorem is generalized for composite systems, in which the superposition assumption of inputs to subsystems is not satisfied. Second, the theorem is generalized to establish a.s.i.l. of an invariant set.

3.4.1. Cases in which the Superposition Assumption of Inputs to Subsystems Is Not Satisfied

In Sec. 2.1., we assumed the superposition assumption of inputs to each transfer system in order to obtain (2-1) as the equation of a physical composite system model (cf. p.13). In engineering problems, we frequently experience that this assumption is not saitsfied. Fig. 3.1. shows a typical example of such case. The pitch and roll control system of an air or undersea craft cited in Sec. 2.1. [14] is a special case of this example. In this example, the inputs and the outputs are all scalors. The functions h_{i2} and h_{i4} of (2-5) and (2-7) are given here by

$$h_{i2}(u_i) = \varphi_i(u_i)$$

$$h_{i4}(y_1, \cdots, y_n, t) = \sum_{j=1}^{n} r_{ij} y_j$$

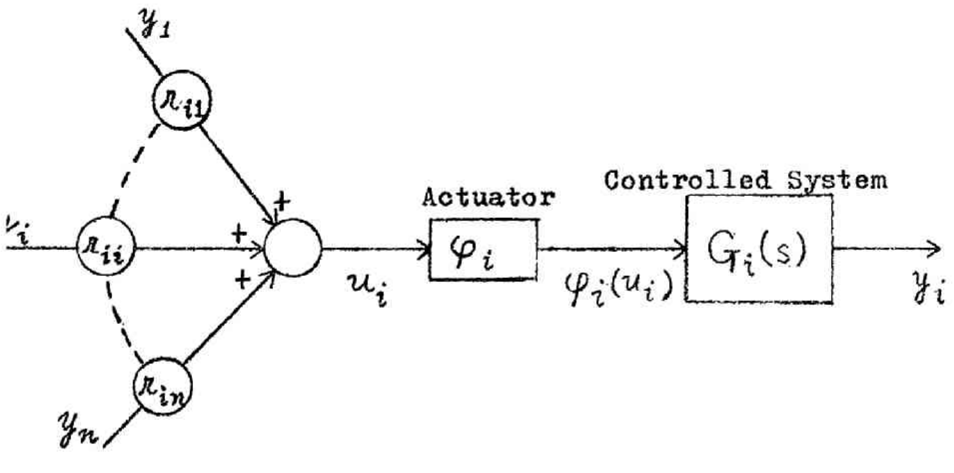


Fig. 3.1. The i-th Transfer System of a Composite System (A case in which the superposition assumption of inputs is not satisfied.)

If $\varphi_i(u_i)$ is a nonlinear function, the equation of the system is not expressed in the form of (2-1). However, if we can assume the inequality

$$0 \leqq \frac{\varphi(u_i)-\varphi(u_i')}{u_i - u_i'} \leqq k_i \quad (u_i \neq u_i') \qquad (3\text{-}38)$$

, we obtain

$$|\varphi_i(u_i)-\varphi_i(r_{ii}y_i)| \leqq \sum_{j;j\neq i} k_i |r_{ij}y_j| \qquad (3\text{-}39)$$

We can generalize Theorem 3.1. for such case, in which an inequalily of the form (3-39) holds, as follows.

Theorem 3.4.

Consider the system given by (2-8) and (2-6), viz. the system given by

$$\frac{dx_i}{dt} = \mathbb{h}_{i1}(x_i,t) + \mathbb{h}_{i5}(\mathbb{h}_{13}(x_1,t),\cdots,\mathbb{h}_{n3}(x_n,t),t \qquad (3\text{-}40)$$

Assume that there are functions $\mathbb{f}_{ij}(x_j,t)$ such that

$$\mathrm{abs}\{\mathbb{h}_{i1}(x_i,t)+\mathbb{h}_{i5}(\mathbb{h}_{i3}(x_1,t),\cdots,\mathbb{h}_{n3}(x_n,t),t)-\mathbb{f}_{ii}(x_i,$$
$$\leqq \sum_{j;j\neq i}^{n} \mathrm{abs}\{\mathbb{f}_{ij}(x_j,t)\} \qquad (3\text{-}41)$$

where $\mathrm{abs}(x)$ means a vector, each component of which is the absolute value of the corresponding component of the vector x , and the inequality between vectors means that the inequality holds for each pair of corresponding components. Assume (2-2) for functions $\mathbb{f}_{ij}(x_j, t)$. Assume (3-i) and (3-ii). Then, the system given by (3-40) is a.s.i.l. if (3-iii).

<u>Proof</u> : The theorem is proved just in the same way as Theorem 3.1. except that the equation (3-25) in the proof of Lemma 3.3. must be replaced by the inequality (3-45) derived in the following.

Put

$$\mathbb{h}_{i1}(x_i, t) + \mathbb{h}_{i5}(\mathbb{h}_{13}(x_1, t), \cdots, \mathbb{h}_{n3}(x_n, t), t) = \mathbb{f}_{ii}(x_i, t) + \mathbb{z}_i \tag{3-42}$$

Then

$$\left(\frac{dv_i}{dt}\right)_{(3-40)} = \left(\frac{dv_i}{dt}\right)_{(2-3-i)} + (\nabla_i v_i)^t \, \mathbb{z}_i \leqq \left(\frac{dv_i}{dt}\right)_{(2-3-i)} + abs(\nabla_i v_i)^t abs(\mathbb{z}_i) \tag{3-43}$$

Here, by (3-41),

$$abs(\mathbb{z}_i) \leqq \sum_{j;j\neq i} abs\{\mathbb{f}_{ij}(x_j, t)\}$$

Therefore,

$$abs(\nabla_i v_i)^t abs(\mathbb{z}_i) \leqq \sum_{j;j\neq i} \|\nabla_i v_i\| \cdot \|\mathbb{f}_{ij}(x_j, t)\| \tag{3-44}$$

Therefore, we obtain

$$\left(\frac{dV}{dt}\right)_{(3-40)} \leq \sum_i w_i \left\{\left(\frac{dv_i}{dt}\right)_{(2-3-i)} + \sum_{j;j\neq i} \|\nabla_i v_i\| \, \|f_{ij}(x_j, t)\|\right\} \tag{3-45}$$

(Q.E.D.)

3.4.2. Asymptotic Stability in the Large of an Invariant Set

We sometimes encounter a case, in which the equilibrium of the system is not a.s.i.l. but a certain invariant set is stable (cf. p. 170 of [44]). We can prove the next theorem just in the same way as Theorem 3.1.

<u>Theorem</u> 3.5.

Consider the system CCS given by (2-1). Assume (3-i') that, for each isolated subsystem (2-3-i), there exist an invariant set Γ_i and a positive function $v_i(x_i, t)$ with continuous partial derivatives such that*

$$\alpha_i(\|x_i - \Gamma_i\|) \leq v_i(x_i, t) \leq \beta_i(\|x_i - \Gamma_i\|) \tag{3-1'}$$

and (3-2) and (3-3) hold for $x_i \notin \Gamma_i$.

Assume (3-ii) for $x_j \notin \Gamma_j$. Then, the set $\widetilde{\Gamma}$ is an invariant set of the system CCS and it is a.s.i.l. if (3-iii).

---

* $\|x_i - \Gamma_i\|$ means the distance of a point x_i from a set Γ_i defined by $\|x_i - \Gamma_i\| = \inf_{y_i \in \Gamma_i} \|x_i - y_i\|$.

Here, $\tilde{\Gamma}$ is the closure of the set $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ (the direct product of $\Gamma_1, \cdots, \Gamma_n$).

Sec. 3.5. Estimate of Transient Behavior

In this section, we make an estimate of the form

$$\|x(t)\| \leqq K \|x(t_0)\| \exp\{-\xi(t-t_0)\} \qquad t \geqq t_0$$

for the transient behavior of the continuous-time composite system CCS.

Theorem 3.6.

Consider the system CCS given by (2-1). Assume (3-vii) that (3-i) holds where eq. (3-1) is replaced by

$$\alpha_i \|x_i\|^2 \leqq v_i(x_i, t) \leqq \beta_i \|x_i\|^2 \qquad (3\text{-}1\text{-}a)$$

Here, α_i and β_i are positive constants.
Assume (3-ii) and (3-iii). Let $C=(c_{ij})$ a matrix given by

$$c_{ii} = \frac{\gamma_i}{2\beta_i} \quad , \quad c_{ij} = -\frac{\delta_i}{2\beta_i}\varepsilon_{ij} \quad (i \neq j) \qquad (3\text{-}46)$$

The matrix C is an M-matrix by (3-iii). (cf. Theorem A.3. in Appendix). Let λ_C the minimum characteristic root of C (cf. Definition A.3. in Appendix).

Then, the solution $x(t)$ of the system CCS is estimated by

$$\|x(t)\| \leq K\|x(t_0)\| \exp\{-(\lambda_c-\varepsilon)(t-t_0)\}$$
$$t \geq t_0 \qquad (3\text{-}47)$$

where ε is an arbitrary positive constant, and K is a positive constant depending upon $\alpha_i, \beta_i, \delta_i, \gamma_i, \varepsilon_{ij}$ and ε.

In the above theorem, the assumption (3-vii) implies the assumption (3-i). Therefore, Theorem 3.1. can assure a.s.i.l. of the system CCS under the assumptions (3-vii), (3-ii) and (3-iii). The above theorem presents more detailed information; i.e. it establishes that the system CCS is exponentially stable* and that an estimate of the damping coefficient is given as the minimum characteristic root of the M-matrix C.

The assumption (3-vii) is just what is assumed in Bailey's theorem (Theorem 3.2.), and it is a necessary and sufficient condition that the isolated subsystem be

---

* A continuous-time dynamical system is said exponentially stable if and only if there are positive numbers ξ and K such that

$$\|x(t)\| \leq K\|x(t_0)\| \exp\{-\xi(t-t_0)\} \qquad t \geq t_0$$

holds for the solution of the system.

exponentially stable. (cf. p. 61 of Krasovskii [45])
In reality, if (3-vii), we can obtain an estimate

$$\|x_i(t)\| \leq K_i \|x_i(t_0)\| \exp\{-\zeta_i(t - t_0)\} \qquad t \geq t_0 \tag{3-48}$$

for the solution $x_i(t)$ of the isolated i-th subsystem.
Here, ζ_i and K_i are positive constants given by

$$\zeta_i = \frac{\gamma_i}{2\beta_i}(-c_{ii}), \quad K_i = \sqrt{\frac{\beta_i}{\alpha_i}} \tag{3-49}$$

The relation (3-48) is easily verified by the relation

$$\left(\frac{dv_i}{dt}\right)_{(2-3-i)} \leq -\frac{\gamma_i}{\beta_i} v_i \tag{3-50}$$

which is obtained from (3-1-a) and (3-2). We can call ζ_i the <u>index of stability</u> of the i-th subsystem.

Now, suppose that each function $v_i(x_i, t)$ is given as a quadratic form of x_i; i.e.

$$v_i(x_i, t) = x_i^t P_i x_i \tag{3-51}$$

where P_i is a positive definite m_i-th order matrix.
The constants β_i and α_i are given as the maximum and the minimum characteristic roots of P_i, respectively.
Since

$$\nabla_i v_i = 2 P_i x_i$$

we obtain

$$\delta_i = 2\beta_i$$

Then, the matrix C is given by

$$c_{ii} = \delta_i \ , \quad c_{ij} = -\varepsilon_{ij} \quad (i \neq j) \tag{3-52}$$

Here, we obtain that, for $v_i(x_i, t)$ given by (3-51), the matrix C is equal to the matrix A of Theorem 3.1. This helps intuitive comprehension of Theorem 3.1. (cf. p. 19) and Theorem 3.6. The next example clarifies the analogy between Theorem 3.6. and the usual results on linear, time-invariant systems.

Consider again the simplest linear composite system with positive connections given by (3-6), which was treated in Sec. 3.1. By putting $v_i = x_i^2$, we obtain (3-9) and

$$\alpha_i = \beta_i = 1$$

Then, the matrix C is given by

$$C = (-\ell_{ij})$$

Then, the assertion of Theorem 3.6. turns out "the solution $x(t)$ of the linear system is estimated by (3-47) where $-\lambda_c$ is the maximum real part of the characteristic roots of the coefficients matrix (ℓ_{ij})." This is the well known result on the linear, time-invariant system (for instance, cf. Chapter 11 of Bellman [22]).

Before proceeding to the proof of Theorem 3.6., let us prove the next lemma.

Lemma 3.5.

Let $A=(a_{ij})$ an n-th order M-matrix, and let λ_A the minimum characteristic root of A Let $C=(c_{ij})$ a matrix given by

$$C = A - \mu I \tag{3-53}$$

where μ is a scalor. The matrix C is an M-matrix if and only if

$$\mu < \lambda_A \tag{3-54}$$

Proof : Choose a sufficiently large scalor ρ such that the matrix $B=(b_{ij})$ given by

$$B = \rho I - A \tag{3-55}$$

is a non-negative matrix. The dominant characteristic root λ_B of B (cf. Definition A.2. in Appendix) is associated with λ_A by

$$\lambda_B = \rho - \lambda_A \tag{3-56}$$

and the matrix C is expressed as

$$C = (\rho - \mu) I - B$$

Therefore, by Theorem A.4., C is an M-matrix if and only if

$$\rho - \mu > \lambda_B$$

Therefore, C is an M-matrix if and only if (3-54).

(Q.E.D.)

Now, let us prove Theorem 3.6.

Proof of Theorem 3.6. : Put

$$\xi = \lambda_C - \varepsilon$$

By Lemma 3.5. the matrix $C - \xi I$ is an M-matrix. Then, by Theorem A.3. in Appendix, the matrix given by

$$C' = (C - \xi I)\, diag\,(2\beta_1, \cdots, 2\beta_n)$$

is an M-matrix. Here, the elements c'_{ij} are expressed as

$$\left.\begin{array}{ll} c'_{ii} = \gamma_i - 2\xi\beta_i & \\ c'_{ij} = -\delta_i\,\varepsilon_{ij} & i \neq j \end{array}\right\} \tag{3-57}$$

By Lemma 3.2., there is a set of positive numbers $w_1, \cdots, w_n$ such that the matrix $B' = (b'_{ij})$ given by

$$B' = \frac{1}{2}\left\{ diag\,(w_1, \cdots, w_n)\, C' + C'^{t}\, diag\,(w_1, \cdots, w_n) \right\}$$

(5-58)

is positive definite. The elements b'_{ij} are expressed as

$$\left.\begin{aligned} b'_{ii} &= w_i \gamma_i - 2\zeta w_i \beta_i \\ b'_{ij} &= -\frac{1}{2}\{w_i \delta_i \varepsilon_{ij} + w_j \delta_j \varepsilon_{ji}\} \quad i \neq j \end{aligned}\right\} \quad (3\text{-}59)$$

Using the above set of $w_1, \cdots, w_n$, put (cf. Lemma 3.3. and its proof)

$$\left.\begin{aligned} b_{ii} &= w_i \gamma_i \\ b_{ij} &= -\frac{1}{2}\{w_i \delta_i \varepsilon_{ij} + w_j \delta_j \varepsilon_{ji}\} \quad i \neq j \end{aligned}\right\} \quad (3\text{-}22)$$

$$V(x, t) = \sum_i w_i v_i(x_i, t) \qquad (3\text{-}23)$$

Then, we obtain

$$\left.\begin{aligned} b_{ii} &= b'_{ii} + 2\zeta w_i \beta_i \\ b_{ij} &= b'_{ij} \qquad\qquad i \neq j \end{aligned}\right\} \quad (3\text{-}60)$$

Since (3-2), (3-3) and (3-4) are also assumed here, the relation (3-27) is obtained just in the same way as the proof of Lemma 3.3. By substituting (3-60) into (3-27), we obtain

$$\left(\frac{dV}{dt}\right)_{(2\text{-}1)} \leqq -\sum_{i,j} b'_{ij} \|x_i\| \cdot \|x_j\| - 2\zeta \sum_i w_i \beta_i \|x_i\|^2 \qquad (3\text{-}61)$$

Since B' is positive definite, we obtain

$$\left(\frac{dV}{dt}\right)_{(2\text{-}1)} \leqq -2\zeta \sum_i w_i \beta_i \|x_i\|^2 \qquad (3\text{-}62)$$

By (3-1-a), we obtain

$$\left(\frac{dV}{dt}\right)_{(2-1)} \leq -2\xi V \tag{3-63}$$

Therefore, we obtain

$$V(x(t),t) \leq V(x(t_0),t_0)\exp\{-2\xi(t-t_0)\}$$
$$t \geq t_0 \tag{3-64}$$

for the solution of the system CCS. By (3-1-a), there are positive constants α and β such that

$$\alpha \|x\|^2 \leq V(x,t) \leq \beta \|x\|^2 \tag{3-65}$$

By (3-64) and (3-65), we obtain the estimate of the solution of the system CCS as

$$\|x(t)\| \leq K \|x(t_0)\| \exp\{-\xi(t-t_0)\} \qquad t \geq t_0$$

Here, K is given by

$$K = \sqrt{\frac{\beta}{\alpha}}$$

The constants α and β depend upon α_i , β_i and w_i . The constants w_i depend upon, by (3-57), γ_i , δ_i , ε_{ij} , β_i and $\xi = \lambda_c - \varepsilon$. Therefore, K depends upon α_i , β_i , γ_i , δ_i , ε_{ij} and ε . (Q.E.D.)

Sec. 3.6. Examples

Two examples are given here. The first example is a system with a single feedback loop. The second is the system treated by Piontkovskii and Rutkovskaya [18].

Example 3.1.

Consider a system given by

$$\left.\begin{aligned} \frac{dx_1}{dt} &= f_{11}(x_1, t) + C_1 x_n \\ \frac{dx_i}{dt} &= f_{ii}(x_i, t) + C_i x_{i-1} \quad i = 2, \cdots, n \end{aligned}\right\} \quad (3\text{-}66)$$

where C_1 and C_i are $m_1 \times m_n$ and $m_i \times m_{i-1}$ matrixes, respectively. The block diagram of this system is given in Fig. 3.2. Here, assume (3-vii). Since the connections are given by C_i, (3-ii) is satisfied where ε_{ij} are

$$\varepsilon_{1n} = \|C_1\|, \quad \varepsilon_{i-1\,i} = \|C_i\| \quad (i = 2, \cdots, n)$$

$$\varepsilon_{ij} = 0 \qquad \text{for the other pairs of } i \text{ and } j \text{ ; } i \neq j$$

Therefore, the matrix A of (3-iii) is given by

$$A = \begin{pmatrix} \frac{\gamma_1}{\delta_1} & 0 & 0 & \cdots & 0 & \|C_1\| \\ \|C_2\| & \frac{\gamma_2}{\delta_2} & 0 & \cdots & 0 & 0 \\ 0 & \|C_3\| & \frac{\gamma_3}{\delta_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \|C_n\| & \frac{\gamma_n}{\delta_n} \end{pmatrix}$$

Therefore, by Theorem 3.1., the system is a.s.i.l. if

$$\frac{\delta_1 \|C_1\|}{\gamma_1} \times \cdots \times \frac{\delta_n \|C_n\|}{\gamma_n} < 1 \qquad (3\text{-}67)$$

and by Theorem 3.6., the estimate of the damping coefficient is given as the minus of the greatest root of

$$(\lambda+\zeta_1)\cdots\cdots(\lambda+\zeta_n) - k_1\cdots\cdots k_n = 0 \qquad (3\text{-}68)$$

$$\zeta_i = \frac{\gamma_i}{2\beta_i}, \quad k_i = \frac{\delta_i \|c_i\|}{2\beta_i}$$

Fig. 3.2. The Block Diagram of Example 3.1.

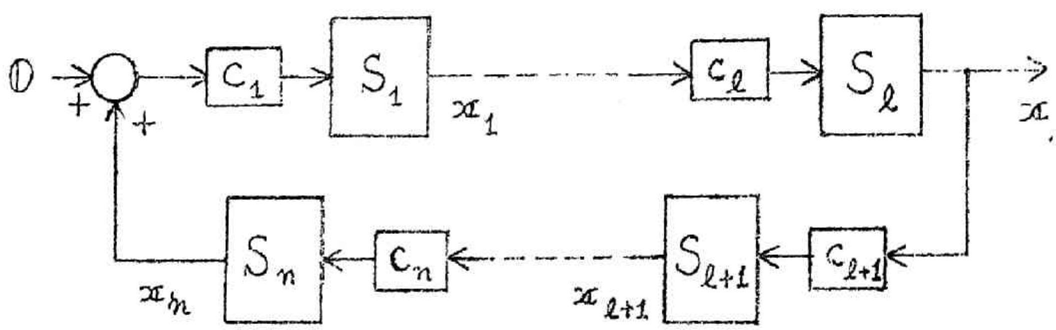
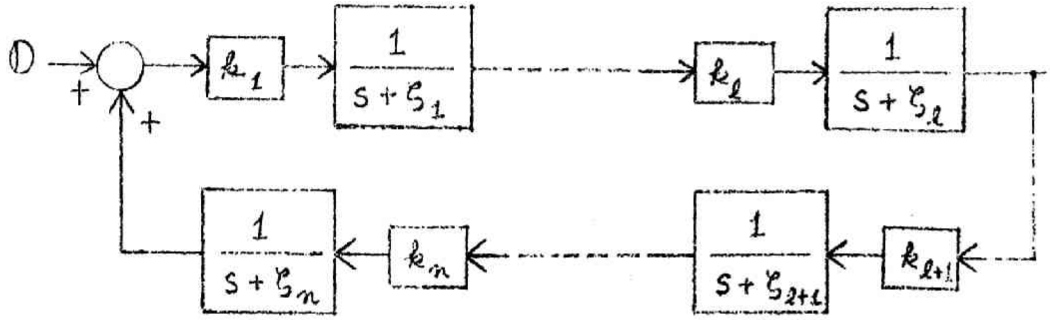


Fig. 3.3. The Linear System Which Gives the Estimate of the Transient Behavior of Example 3.1.



If we regard δ_i / γ_i as a gain of the subsystem, (3-67) means the condition that the loop gain is less than 1. Eq. (3-68) is the characteristic equation of the linear system given in Fig. 3.3.

Example 3.2.

Consider a system given by

$$\left.\begin{aligned} \frac{dx_i}{dt} &= -\rho_i x_i + \sigma \qquad\qquad i = 1, 2, 3, 4 \\ \frac{d\sigma}{dt} &= \sum_{i=1}^{4} \beta_i x_i + \lambda p_2 \sigma - f(\sigma) \end{aligned}\right\} \quad (3\text{-}69)$$

$$\rho_4 \geqq \rho_3 \geqq \rho_2 \geqq \rho_1 > 0$$

$$\lambda > 0 \;, \quad p_2 < 0$$

$$f(0) = 0 \;, \quad \sigma f(\sigma) > 0$$

The block diagram of this system is given in Fig. 3.4. This system was treated by Piontkovskii and Rutkovskaya [18]. They gave

$$|\lambda p_2| > \frac{4}{\rho_1} \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2} = K_1 \quad (3\text{-}70)$$

as a sufficient condition for a.s.i.l. of this system.

This system contains only one nonlinearity. So, we can use the Popov's criterion (cf. Chapter 3 of [3]). The frequency response of the linear part is given by

Fig. 3.4.

The Block Diagram

of Example 3.2.

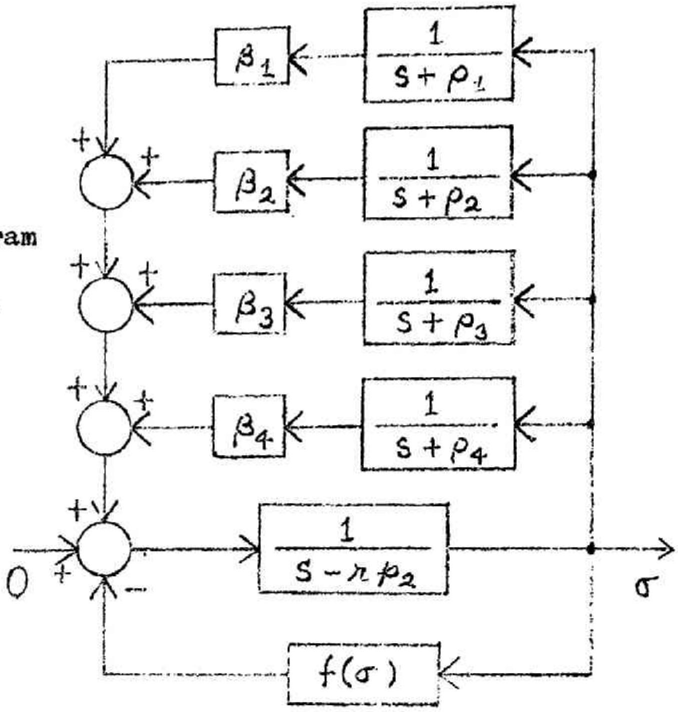
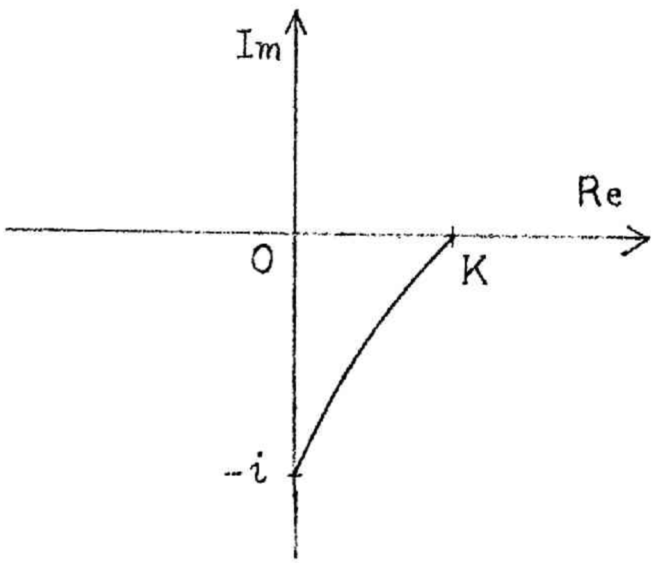


Fig. 3.5. The Modified Frequency Response

of the Linear Part of Example 3.2.



$$G(i\omega) = \left\{ i\omega - r p_2 - \sum_{j=1}^{4} \frac{\beta_j}{i\omega + \rho_j} \right\}$$

The modified frequency response defined by

$$G^*(i\omega) = \mathrm{Re}\{G(i\omega)\} + i\omega\, \mathrm{Im}\{G(i\omega)\}$$

is as drawn in Fig. 3.5. Here, the constant K is given by

$$K = \left\{ - r p_2 - \sum_{j=1}^{4} \frac{\beta_j}{\rho_j} \right\}^{-1} \tag{3-71}$$

Therefore, Popov's theorem gives

$$|r p_2| > \frac{\beta_1}{\rho_1} + \frac{\beta_2}{\rho_2} + \frac{\beta_3}{\rho_3} + \frac{\beta_4}{\rho_4} \equiv K_2 \tag{3-72}$$

as a sufficient condition for a.s.i.l. of this system.

Here, let us apply Theorem 3.1. and Theorem 3.6. to this system. Let us regard this system as composed of five subsystems, each of which is described by one of the differential equations (3-69). By putting

$$v_i = x_i^{\,2} \quad (i = 1, 2, 3, 4), \quad v_5 = \sigma^2$$

the assumption (3-vii) is satisfied and the constants are given by

$$\alpha_i = \beta_i = 1 \qquad i = 1, 2, 3, 4, 5$$
$$\gamma_i = 2\rho_i, \quad \delta_i = 2 \qquad i = 1, 2, 3, 4$$
$$\gamma_5 = 2|r p_2|, \quad \delta_5 = 2$$

$$\varepsilon_{ij} = 0 \qquad\qquad i, j = 1, 2, 3, 4$$

$$\varepsilon_{i5} = 1, \quad \varepsilon_{5i} = |\beta_i| \qquad i = 1, 2, 3, 4$$

Therefore, the matrix A and C are given by

$$A = C = \begin{pmatrix} \rho_1 & 0 & 0 & 0 & -1 \\ 0 & \rho_2 & 0 & 0 & -1 \\ 0 & 0 & \rho_3 & 0 & -1 \\ 0 & 0 & 0 & \rho_4 & -1 \\ -|\beta_1| & -|\beta_2| & -|\beta_3| & -|\beta_4| & |r\rho_2| \end{pmatrix} \tag{3-73}$$

Therefore, we obtain

$$|r\rho_2| > \frac{|\beta_1|}{\rho_1} + \frac{|\beta_2|}{\rho_2} + \frac{|\beta_3|}{\rho_3} + \frac{|\beta_4|}{\rho_4} \equiv K_3 \tag{3-74}$$

as a sufficient condition for a.s.i.l. of the system.

Now, let us compare the three stability conditions. It is easy to show

$$K_1 > K_3 \geqq K_2 \quad \text{for} \quad \beta_i \neq 0 \tag{3-75}$$

Therefore, our condition (3-74) is between Popov's condition and Piontkovskii-Rutkovskaya's condition. The difference of K_3 and K_2 shows an essential character of Theorem 3.1., viz. the fact that we only consider the absolute value of the gain of the connection between subsystems and neglect its phase-shifting effect (cf. the assumption (3-ii)).

The characteristic equation of the matrix C is given as

$$
\begin{aligned}
&|\lambda \rho_2| (\lambda-\rho_1)(\lambda-\rho_2)(\lambda-\rho_3)(\lambda-\rho_4) \\
&\quad + |\beta_1| (\lambda-\rho_2)(\lambda-\rho_3)(\lambda-\rho_4) \\
&\quad + |\beta_2| (\lambda-\rho_1)(\lambda-\rho_3)(\lambda-\rho_4) \\
&\quad + |\beta_3| (\lambda-\rho_1)(\lambda-\rho_2)(\lambda-\rho_4) \\
&\quad + |\beta_4| (\lambda-\rho_1)(\lambda-\rho_2)(\lambda-\rho_3) = 0
\end{aligned}
\tag{3-76}
$$

This equation gives the estimate of the damping coefficient of the system.

Chapter 4 Asymptotic Stability in the Large of Sampled-Data Composite Systems

Sec. 4.1. Main Theorem

Concerning a.s.i.l. of the sampled-data composite system SCS given by (2-9), we have the next theorem.

Theorem 4.1.

Consider the system SCS given by (2-9). Assume

(4-i) that, for each isolated subsystem (2-11-i), there exists a norm $N_i(x_i)$ of the state vector x_i such that

$$N_i(g_i(x_i, \tau)) - N_i(x_i) \leq -\gamma_i M_i(x_i) \tag{4-1}$$

where γ_i is a positive constant and $M_i(x_i)$ is a non-negative valued function of x_i satisfying

$$M_i(\mathbb{0}) = 0 \tag{4-2}$$

and

$$\inf_{x_i \in \Delta_i} M_i(x_i) > 0 \tag{4-3}$$

for any bounded, closed subset Δ_i of the vector space* X_i which does not contain the origin, (4-ii) and that there are non-negative constants

ε_{ij} $(i \neq j)$ such that

$$N_i(g_{ij}(x_j, \tau)) \leq \varepsilon_{ij} M_j(x_j) \quad i \neq j \qquad (4\text{-}4)$$

The system SCS is a.s.i.l. if

(4-iii) the matrix $A = (a_{ij})$ is an M-matrix, i.e. the leading principal minor determinants of A are all positive (cf. Definition A.1. in Appendix). Here, a_{ij} 's are given by

$$a_{ii} = \gamma_i \ , \ a_{ij} = -\varepsilon_{ij} \quad (i \neq j) \qquad (4\text{-}5)$$

The theorem is proved in the next section. The assumption (4-i) means that the function $g_{ii}(x_i, \tau)$ is a contraction with regard to the norm N_i. If each subsystem is low-dimensional and contains only a few nonlinearities, such a norm is obtained by the results of previous researches [9, 26-28, 46]. If a Lyapunov function of a quadratic form of the state vector is constructed for the isolated subsystem, the norm defined as the root of the quadratic form satisfies (4-i).

---

* X_i is the m_i-dimensional vector space to which x_i belongs.

So, the previous researches on the construction of a Lyapunov function for sampled-data systems [47-49] also contribute to establishment of the assumption (4-i). The constant ε_{ij} in the assumption (4-ii) is the upper bound of the d.c. gain of the connection with regard to the norm $N_i(x_i)$ and the function $M_j(x_j)$.

In the stability condition (4-iii), a_{ii} is a kind of <u>index</u> <u>of</u> <u>stability</u> of each subsystem and $|a_{ij}|$ $(i \neq j)$ is the strength of the connection. Therefore, we obtain the same interpretation of the theorem as the continuous-time case; viz. the composite system is stable if the stability of subsystems is superior to the interconnection among subsystems.

Now, let us see that the stability condition of Theorem 4.1. is a necessary and sufficient condition for a.s.i.l. of a certain linear system. Consider a linear difference equation with positive coefficients given by

$$x_i(\tau+1) = \sum_{j=1}^{n} \ell_{ij}\, x_j(\tau) \qquad i = 1, \cdots, n \quad (4\text{-}6)$$

where ℓ_{ij} are constants satisfying

$$0 \leqq \ell_{ii} < 1\,, \quad \ell_{ij} \geqq 0 \quad (i \neq j) \qquad (4\text{-}7)$$

The above equation can be interpreted as describing a composite system of n subsystems. In parallel with the continuous-time case, we call this system the simplest sampled-data linear composite system with positive connections. Now, let us apply Theorem 4.1. to the system given by (4-6). The isolated i-th subsystem is given by

$$x_i(\tau+1) = \ell_{ii}\, x_i(\tau) \tag{4-8-i}$$

Here, by putting $N_i(x_i) = |x_i|$, the assumptions (4-i) and (4-ii) are satisfied, where

$$\begin{aligned} M_i(x_i) &= |x_i| \\ \gamma_i = 1 - \ell_{ii} \ , \quad \varepsilon_{ij} &= \ell_{ij} \quad (i \neq j) \end{aligned} \tag{4-9}$$

Therefore, we obtain

$$A = I - L \tag{4-10}$$

where L is the coefficient matrix (ℓ_{ij}). Therefore, the condition (4-iii) of Theorem 4.1. is, by the condition (A-x) of Theorem A.4. in Appendix, equivalent that the dominant characteristic root of the positive matrix L (cf. Definition A.2.) is less than 1. By the condition (A-viii) of Theorem A.4., this is necessary and sufficient for a.s.i.l. of the linear system (4-6). Thus, we obtain that (4-iii) is necessary and sufficient for a.s.i.l. of the linear system (4-6).

Sec. 4.2. Proof of the Main Theorem and a Supplementary Theorem

Here, Theorem 4.1. is proved and a remark on the condition (4-i) is made.

First, let us prove the next lemma.

Lemma 4.1.

Consider the system SCS given by (2-9). Assume (4-i) and (4-ii). The system SCS is a.s.i.l. if

(4-iv) there exists a set of n positive numbers w_1, ---, w_n such that

$$\sum_i a_{ij} w_j \equiv d_j > 0 \tag{4-11}$$

Here a_{ij} are the constants given by (4-5).

Proof : From (2-9), we obtain*

$$\begin{aligned} &\{N_i(x_i(\tau+1)) - N_i(x_i(\tau))\}_{(2-9)} \\ &\leqq N_i\{g_{ii}(x_i(\tau),\tau)\} - N_i(x_i(\tau)) \\ &\qquad + \sum_{j;j\neq i} N_i\{g_{ij}(x_j(\tau),\tau)\} \end{aligned} \tag{4-12}$$

---

* $\{N_i(x_i(\tau+1)) - N_i(x_i(\tau))\}_{(2-9)}$ is the difference of $N_i(x_i(\tau+1))$ and $N_i(x_i(\tau))$ along the solution of (2-

By (4-1) and (4-4), we obtain

$$\{N_i(x_i(\tau+1)) - N_i(x_i(\tau))\}_{(2-7)}$$

$$\leqq -\gamma_i M_i(x_i(\tau)) + \sum_{j;j\neq i} \varepsilon_{ij} M_j(x_j(\tau)) \tag{4-13}$$

Here, using the set of $w_1, \cdots, w_n$ given in (4-iv), put

$$N(x) = \sum_i w_i N_i(x_i) \tag{4-14}$$

Then, $N(x)$ is a norm of the vector x. By (4-13), we obtain

$$\{N(x(\tau+1)) - N(x(\tau))\}_{(2-7)}$$

$$\leqq -\sum_{i,j} w_i a_{ij} M_j(x_j(\tau))$$

$$= -\sum_j d_j M_j(x_j(\tau)) \tag{4-15}$$

By the property of $M_j(x_j)$ given in (4-i), we obtain

$$\inf_{x\in\Delta} \sum_j d_j M_j(x_j) > 0 \tag{4-16}$$

for any bounded, closed subset Δ of X which does not contain the origin. Therefore, the system SCS is a.s.i.l. (cf. Theorem 1 at p. 396 of [9]) (Q.E.D.)

By the condition (A-iv) of Theorem A.1. in Appendix, the condition (4-iv) is equivalent to the condition (4-iii).

Therefore, we obtain Theorem 4.1.

Now, let us consider the requirement that $N_i(x_i)$ be a norm of x_i. A norm of a vector x_i is defined as a real-valued function $N_i(x_i)$ of x_i satisfying

$$N_i(x_i) > 0 \qquad x_i \neq 0 \tag{4-17}$$

$$N_i(x_i) = 0 \qquad x_i = 0 \tag{4-18}$$

$$N_i(\lambda x_i) = |\lambda|\, N_i(x_i) \tag{4-19}$$

$$N_i(x_i + x_i') \leq N_i(x_i) + N_i(x_i') \tag{4-20}$$

where λ is an arbitrary real number. In the proof of Lemma 4.1., the property given by (4-20) is used to derive (4-12) from (2-9), and the property given by (4-17) and (4-18) is used to establish the same property for $N(x)$, which is necessary to conclude a.s.i.l. of the system SCS from the relation (4-15) and (4-16). However, the linearity property given by (4-19) is not used anywhere in the proof. Therefore, we have the next theorem*.

---

* The main theorem is stated here requiring N_i be a norm 1) because we can have a simpler expression by this, and 2) because we find many literatures which give a norm satisfying the condition of (4-i). Even if we remove the requirement (4-19), we hardly have more possibility of establishing (4-i).

Theorem 4.2.

In Theorem 4.1., Lemma 4.1., and Theorem 4.3.-4.6., we can replace the requirement that $N_i(x_i)$ be a norm by the requirement that $N_i(x_i)$ be a real-valued function of x_i satisfying (4-17), (4-18) and (4-20).

As an example of a real-valued function $N_i(x_i)$ of a vector $x_i = (x_{i1}, \cdots, x_{im})^t$ satisfying (4-17), (4-18) and (4-20), we can give

$$N_i(x_i) = \sqrt{|x_{i1}| + \cdots + |x_{im}|}$$

Sec. 4.3. Generalization of the Main Theorem

First, an intuitive interpretation of Lemma 4.1. is given in 4.3.1. With relation to this interpretation, Theorem 4.1. and Lemma 4.1. are generalized. In 4.3.2. and 4.3.3., we give two generalized forms of Theorem 4.1., which correspond respectively to Theorem 3.4. and to Theorem 3.5. in continuous-time case.

4.3.1. An Interpretation of Lemma 4.1. and Generalization of Theorem 4.1. and Lemma 4.1.

Generally, the assertion that a system is a.s.i.l. means that the state of the system is restored to the equilibrium wherever it starts. In Lemma 4.1., the assumption (4-i) means that the restoration of the isolated subsystem during one sampling period amounts to, at least, $\gamma_i M_i(x_i(\tau))$ in some measure N_i. Since the subsystems are interconnected with each other, the restoring movement of one subsystem causes disturbances in the other subsystems. The assumption (4-ii) means this disturbance acting on another subsystem during one sampling period is less than $\varepsilon_{ji} M_i(x_i(\tau))$. The condition (4-iii) of Lemma 4.1. means that, if we weight subsystems appropriately, the amount of the all disturbances caused by one subsystem is less than its own restoration.

The above interpretation gives an important result. Consider a case in which the movement of the system during one sampling period depends only upon the outputs z_1, ---, z_n of n controllers, i.e. the values of functions $g_{ij}(x_j, \tau)$ are determined only by $z_1, ---, z_n$. (cf. Fig.4.1.) This is the case if the sampling period is sufficiently long compared with the time constants of the controlled system. Here, assume

$$g_{ii}(x_i, \tau) = x_i \quad , \quad g_{ij}(x_i, \tau) = 0 \quad (i \neq j)$$

for

$$z_j = 0 \qquad j = 1, ---, n$$

This condition is satisfied if the controlled system

Fig. 4.1. A Sampled-Data Composite System

(Each controller corresponds to a subsystem)

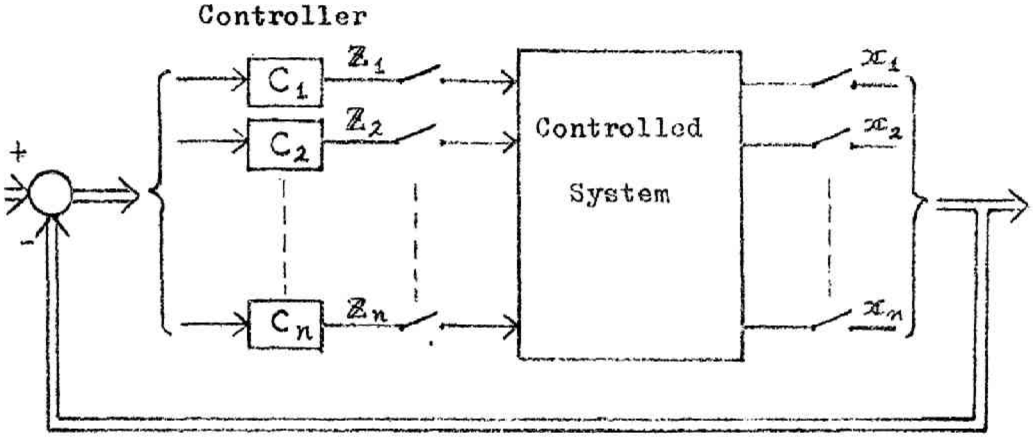
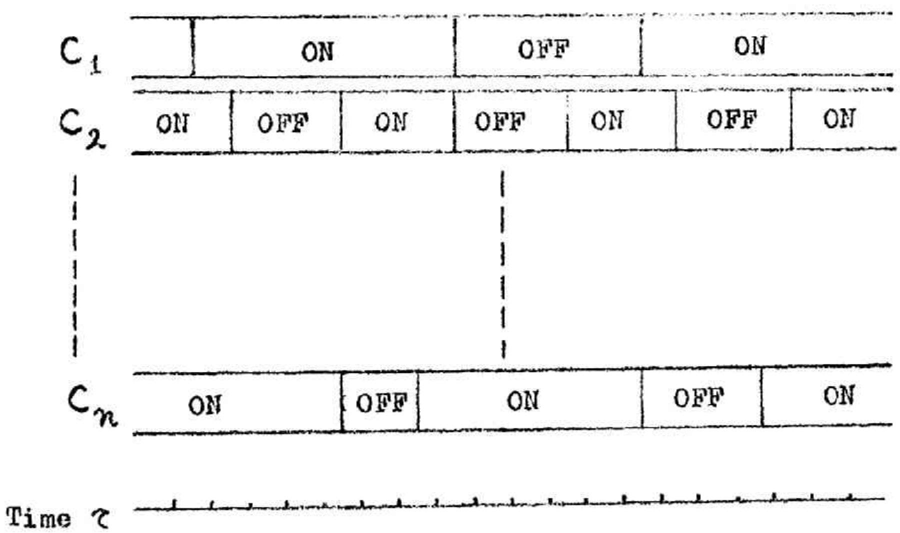


Fig. 4.2. Operation of Controllers



contains integrators appropriately. Under those assumptions, consider the case in which the i-th controller does not work at some sampling instants but works at least once in p_i sampling periods. This situation is explained in Fig. 4.2. Even in such a case, the assertion of Theorem 4. remains valid. Let us state this result as a theorem in a little extended form.

Theorem 4.3.

Consider the system SCS given by (2-9). Assume that, for each subsystem, there is a set of integers

$$\Sigma_i = \{ \sigma_k^{(i)} \ ; \ k = 0, \pm 1, \pm 2, \cdots \}$$

such that at least one $\sigma_k^{(i)}$ is contained in any interval $[\tau, \tau + p_i)$ $(\tau = 0, 1, 2, \cdots)$ where p_i is a constant positive integer. Assume that (4-i) holds for $\tau \in \Sigma$ and

$$\left. \begin{aligned} g_{ii}(x_i, \tau) &= x_i \\ g_{ji}(x_i, \tau) &= 0 \qquad j \neq i \end{aligned} \right\} \text{ for } \tau \notin \Sigma_i \tag{4-21}$$

Assume (4-ii).

Then, the system SCS is a.s.i.l. if (4-iii).

The above theorem can be easily proved by slightly changing the proof of Theorem 4.1. In the theorem, each subsystem can be regarded as corresponding to one controller. The set Σ_i gives the sampling instants at which the controller works. This theorem is useful when we study a case in which the sampling periods of subsystems are not same but their ratios are rational numbers.

In relation to the above consideration, we can extend Lemma 4.1. for the case in which there are two origines of the movement of each subsystem.

<u>Theorem</u> 4.4.

Consider the system SCS given by (2-9). Assume (4-v) that, for each isolated subsystem, there exists a norm $N_i(x_i)$ of the state vector x_i such that

$$N_i(g_i(x_i,\tau)) - N_i(x_i) \leqq -\gamma_i^{(1)} M_i^{(1)}(x_i) - \gamma_i^{(2)} M_i^{(2)}(x_i) \qquad (4\text{-}22)$$

where $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ are positive constants and $M_i^{(1)}(x_i)$ and $M_i^{(2)}(x_i)$ are scalor functions of x_i satisfying

$$M_i^{(1)}(0) = M_i^{(2)}(0) = 0 \ , \ M_i^{(2)}(x_i) \geqq 0 \qquad (4\text{-}23)$$

and

$$\inf_{x_i \in \Delta_i} M_i^{(1)}(x_i) > 0 \qquad (4\text{-}24)$$

for any bounded, closed subset Δ_i of X_i which does not contain the origin,

(4-vi) and that there are non-negative constants $\varepsilon_{ij}^{(1)}$ and $\varepsilon_{ij}^{(2)}$ $(i \neq j)$ such that

$$N_i(g_{ij}(x_j, \tau)) \leqq \varepsilon_{ij}^{(1)} M_j^{(1)}(x_j) + \varepsilon_{ij}^{(2)} M_j^{(2)}(x_j) \tag{4-25}$$

The system SCS is a.s.i.l. if

(4-vii) there exists a set of n positive numbers $w_1, \cdots$ $\cdots, w_n$ such that

$$\sum_i a_{ij}^{(1)} w_i \equiv d_j^{(1)} > 0 \tag{4-26}$$

$$\sum_i a_{ij}^{(2)} w_i \equiv d_j^{(2)} \geqq 0 \tag{4-27}$$

Here, $a_{ij}^{(1)}$ and $a_{ij}^{(2)}$ are given by

$$a_{ii}^{(s)} = \gamma_i^{(s)}, \quad a_{ij}^{(s)} = \varepsilon_{ij}^{(s)} \quad (i \neq j) \tag{4-28}$$
$$s = 1, 2$$

The above theorem can be proved just in the same way as Lemma 4.1. At present, we don't have any condition of the form (4-iii), which is equivalent to (4-vii).

4.3.2. Cases in which the Superposition Assumption of Inputs to Subsystems Is Not Satisfied

<u>Theorem</u> 4.5.

Consider the sampled-data system given by

$$x_i(\tau+1) = \mathbb{h}_{i1}(x_i(\tau), \tau) + \mathbb{h}_{i2}(x_1(\tau), \cdots, x_n(\tau), \tau) \tag{4-29}$$

Assume that there are functions $g_{ij}(x_i, \tau)$ which satisfy (2-10), (4-i), (4-ii) and

$$N_i(\mathbb{h}_{i1}(x_i, \tau) + \mathbb{h}_{i2}(x_1, \cdots, x_n, \tau) - g_{ii}(x_i, \tau)) \leqq \sum_{j; j\neq i} N_i(g_{ij}(x_j, \tau)) \tag{4-30}$$

Then, the system (4-29) is a.s.i.l. if (4-iii).

If we refer to the proof of Theorem 4.1., the above theorem is evident.

4.3.3. Asymptotic Stability in the Large of an Invariant Set

<u>Theorem</u> 4.6.

Consider the system SCS given by (2-9) and assume

$$\left.\begin{array}{ll} g_{ii}(x_i, \tau) \in \Gamma_i & x_i \in \Gamma_i \\ g_{ij}(x_j, \tau) = 0 & x_j \in \Gamma_j, \; i \neq j \end{array}\right\} \tag{4-31}$$

where Γ_j is a closed subset of X_j which contains the origin. Then, the set $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is the invariant set of the system SCS. Assume

(4-i') that, for each isolated subsystem (2-11-i), there exists a norm $N_i(x_i)$ of the state vector x_i such that (4-1) holds. Here, γ_i is a positive constant and $M_i(x_i)$ is a non-negative valued function of x_i satisfying (4-2) and (4-3) for any bounded, closed subset Δ_i of X_i which does not contain any point of Γ_i.

Assume (4-ii) for $x_j \notin \Gamma_j$.

Then, the set Γ is a.s.i.l. if (4-iii).

The above theorem corresponds to Theorem 3.4. in continuous-time case, but it differs in the point that $g_{ij}(x_j, \tau) = 0$ is required for $x_j \in \Gamma_j$. This difference is caused by the fact that the state "jumps" in the sampled-data case. The theorem is proved just in the same way as Theorem 4.1.

Sec. 4.4. Estimate of Transient Behavior

In this section, we make an estimate of the form

$$\|x(\tau)\| \leq K \|x(\tau_0)\| \, \eta^{(\tau-\tau_0)} \qquad \tau \geq \tau_0$$

for the transient behavior of the sampled-data composite system SCS.

Theorem 4.7.

Consider the system SCS given by (2-9). Assume (4-i), (4-ii), (4-iii) and

$$M_i(x_i) = N_i(x_i) \tag{4-32}$$

Let λ_A the minimum characteristic root of the matrix A.

Then, the solution $x(\tau)$ of the system SCS is estimated by*

$$\|x(\tau)\| \leq K \|x(\tau_0)\| \, (1-\lambda_A+\varepsilon)^{\tau-\tau_0} \tag{4-33}$$

$$\tau \geq \tau_0$$

where ε is an arbitrary positive constant, and K is a positive constant depending upon the positive constants γ_i, ε_{ij}, ε and the norms N_i .

---

* $\|x\|$ is the Euclidean norm of x . (cf. footnote at p. 17)

This theorem corresponds to Theorem 3.6. in the continuous-time case. The theorem tells that, if we have (4-32) in addition to the three conditions of Theorem 4.1., we can obtain more than the only a.s.i.l. of the system SCS; i.e. we can assert that the system SCS is exponentially stable* and that the decay of the transient response of the system is associated with the characteristic root of the matrix A .

Now, let us apply the above theorem to the simplest sampled-data linear composite system with positive connections given by (4-6). By putting $N_i(x_i) = |x_i|$, we obtain (4-9) and (4-10). By (4-10), we obtain

$$\lambda_L = 1 - \lambda_A$$

where λ_L is the dominant characteristic root of the

---

* A sampled-data dynamical system is said exponentially stable if and only if there are positive numbers η and K such that

$$\|x(\tau)\| \leqq K \|x(\tau_0)\| \; \eta^{(\tau-\tau_0)} \qquad \tau \geqq \tau_0$$

$$0 \leqq \eta < 1$$

holds for the solution of the system.

coefficient matrix $L = (\ell_{ij})$ (cf. Definition A.2. in Appendix). By Theorem A.4., λ_L is the maximum absolute value of the characteristic roots of L. Therefore, the assertion of Theorem 4.7. turns out "the solution of the linear system is estimated by

$$\|x(\tau)\| \leqq K \|x(\tau_0)\| (\lambda_L + \varepsilon)^{(\tau-\tau_0)} \qquad \tau \geqq \tau_0$$

where λ_L is the maximum absolute value of the characteristic roots of the coefficient matrix $L = (\ell_{ij})$."
This is the well known result on the linear, time-invariant sampled-data system.

Now, let us prove Theorem 4.7.

<u>Proof of Theorem 4.7.</u> : First, let us prove

$$0 < \lambda_A \leqq 1 \tag{4-34}$$

By (4-1) and (4-32), we obtain

$$N_i(g_i(x_i, \tau)) \leqq (1 - \gamma_i)\, N_i(x_i) \tag{4-35}$$

Therefore, we obtain

$$\gamma_i \leqq 1 \tag{4-36}$$

Therefore, by Theorem A.5. in Appendix, we obtain (4-34).

Now, put

$$\eta = \lambda_A - \varepsilon < 1 \tag{4-37}$$

By Lemma 3.5., the matrix $C=(c_{ij})$ given by

$$C = A - \eta I$$

is an M-matrix. The elements c_{ij} are related to a_{ij} by

$$a_{ii} = c_{ii} + \eta \quad , \quad a_{ij} = c_{ij} \quad (i \neq j) \tag{4-38}$$

By the condition (A-iv) of Theorem A.1. in Appendix, there is a set of n positive numbers $w_1, ---, w_n$ such that

$$\sum_i c_{ij} w_i \equiv d'_j > 0 \tag{4-39}$$

Using the above set of $w_1, ---, w_n$ put (cf. Proof of Lemma 4.

$$N(x) = \sum_i w_i N_i(x_i) \tag{4-14}$$

Since (4-i) and (4-ii) are also assumed here, we can obtain (4-15) just in the same way as the proof of Lemma 4.1. By substituting the relation (4-38) into (4-15), we obtain

$$\begin{aligned} &\{N(x(\tau+1)) - N(x(\tau))\}_{(2-9)} \\ &\qquad \leqq -\sum_{i,j} w_i c_{ij} N_j(x_j(\tau)) - \sum_i w_i \eta N_i(x_i(\tau \\ &\qquad = -\sum_j d'_j N_j(x_j(\tau)) - \eta N(x(\tau)) \end{aligned} \tag{4-40}$$

By (4-39), we obtain

$$\{N(x(\tau+1))-N(x(\tau))\}_{(2-9)} \leqq -\eta\, N(x(\tau)) \tag{4-41}$$

Therefore, we obtain

$$N(x(\tau+1)) \leqq (1-\eta)\, N(x(\tau)) \tag{4-42}$$

for the solution of the system SCS. Here, note that, by (4-37), we have

$$1-\eta > 0$$

By (4-42), we obtain

$$N(x(\tau)) \leqq N(x(\tau_0))\,(1-\eta)^{(\tau-\tau_0)} \tag{4-43}$$

$$\tau \geqq \tau_0$$

Since $N(x)$ is a norm of x by the definition (4-14), there are positive constants α and β such that

$$\alpha \|x\| \leqq N(x) \leqq \beta \|x\| \tag{4-44}$$

Therefore, we obtain the relation (4-33) from the relation (4-43), where K is given by

$$K = \frac{\beta}{\alpha}$$

The constants α and β are determined by the property of the norms N_i and the constants w_i. The constants w_i depend upon the matrix C, which is determined by the constants γ_i, ε_{ij} and the arbitrary constant ε.

Therefore, K depends upon γ_i , ε_{ij} , ε and N_i .

(Q.E.D.)

Sec. 4.5. Examples

Two examples are given here. The first example is a relay controled sampled-data system composed of one-dimensional subsystems. The second example is a system composed of 2 two-dimensional subsystems.

Example 4.1.

Consider a sampled-data system given by

$$x_i(\tau+1) = x_i(\tau) + \sum_j b_{ij} u_j(\tau) \tag{4-45}$$

$$u_i(\tau) = \varphi_i(-x_i(\tau)) \tag{4-46}$$

Here, b_{ij} are constants satisfying

$$b_{ii} > 0 \tag{4-47}$$

and $\varphi_i(\sigma_i)$ are functions given by

$$\varphi_i(\sigma_i) = \begin{cases} d_i & \sigma_i > c_i \\ 0 & |\sigma_i| \leqq c_i \\ -d_i & \sigma_i < -c_i \end{cases} \tag{4-48}$$

where c_i and d_i are positive constants. These functions describe ideal relays with dead zone. The block diagram of this system is given in Fig. 4.3.

First, let us investigate the isolated i-th subsystem given by

$$x_i(\tau+1) = x_i(\tau) + b_{ii}\,\varphi_i(-x_i(\tau)) \qquad (4\text{-}49\text{-}i)$$

Put

$$\Gamma_i = [-c_i\,,\ c_i]$$

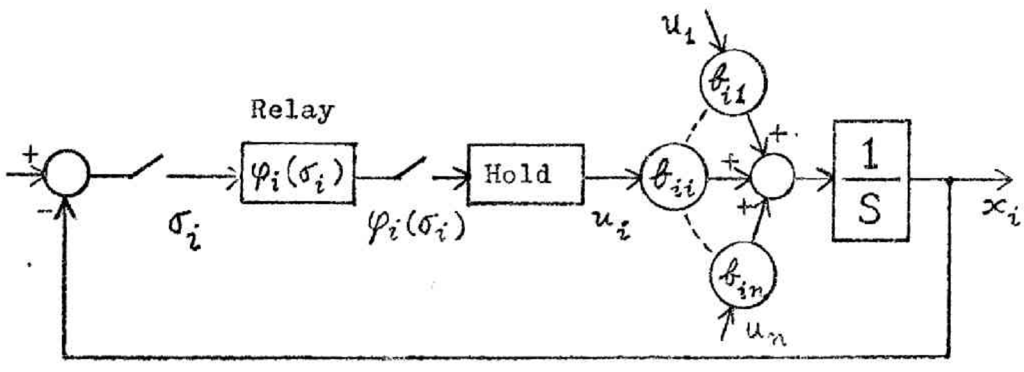
Then, Γ_i is the set of the equilibrium points of the isolated i-th subsystem. If $b_{ii}\,d_i \leqq c_i$, we obtain

$$|x_i+\varphi_i(-x_i)|-|x_i| \leqq -b_{ii}\,|\varphi_i(-x_i)|\,,\quad x_i \notin \Gamma_i \qquad (4\text{-}50)$$

If $c_i < b_{ii}\,d_i < 2\,c_i$, we obtain

$$|x_i+\varphi_i(-x_i)|-|x_i| \leqq -\frac{2c_i-d_i}{d_i}\,b_{ii}\,|\varphi_i(-x_i)|$$
$$x_i \notin \Gamma_i \qquad (4\text{-}51)$$

Fig. 4.3. The Block Diagram of Example 4.1.



Therefore, when $b_{ii} d_i < 2 c_i$, the set Γ_i is a.s.i.l. and the condition (4-i') of Theorem 4.6. is satisfied where

$$N_i(x_i) = |x_i| \quad , \quad M_i(x_i) = |\varphi_i(x_i)|$$

When $b_{ii} d_i \geqq 2 c_i$, we cannot obtain that $|x_i + \varphi_i(-x_i)| - |x_i|$ is always negative. When $b_{ii} d_i > 2 c_i$, self-oscillations are observed if we choose the initial values as

$$c_i < x_i(\tau_0) < b_{ii} d_i - c_i$$

A self-oscillation of this kind is stable but not asymptotically stable because a slight change of the initial value provokes another self-oscillation.

Now, let us investigate the behavior of the composite system given by (4-45) and (4-46). The set Γ given by

$$\Gamma = \{ x \; ; \; |x_i| \leq c_i \}$$

is the direct product of $\Gamma_1, ---, \Gamma_n$ and the set of the equilibrium points of the composite system. If the constants c_i and d_i of the relays are chosen as

$$b_{ii} d_i \geqq 2 c_i \tag{4-52}$$

our theorem given in the preceding sections cannot give any result. If the constants c_i and d_i are chosen as

$$b_{ii} d_i < 2c_i \tag{4-53}$$

the assumptions (3-i') and (3-ii) of Theorem 4.6. are satisfied. Therefore, by Theorem 4.6., the set Γ is a.s.i.l. if the matrix A is an M-matrix, where A is given by

$$\left.\begin{array}{l} a_{ii} = \begin{cases} b_{ii} & b_{ii} d_i \leqq c_i \\ \dfrac{2c_i - d_i}{d_i} b_{ii} & c_i \leqq b_{ii} d_i \leqq 2c_i \end{cases} \\ \\ a_{ij} = -|b_{ij}| \end{array}\right\} \tag{4-54}$$

Here note that

$$a_{ij} \leqq b^{\dagger}_{ij} \tag{4-55}$$

where $b^{\dagger}_{ij}$ are given by

$$b^{\dagger}_{ii} = b_{ii} \quad , \quad b^{\dagger}_{ij} = -|b_{ij}| \quad (i \neq j)$$

This relation means that the matrix $B^{\dagger} = (b^{\dagger}_{ij})$ is always an M-matrix if the matrix A is an M-matrix. This means that, if $B^{\dagger}$ is not an M-matrix, Theorem 4.6. cannot establish a.s.i.l. of the system whatever the constants of the relays may be. At present we don't have any complete answer to the quenstion whether the composite system can be made a.s.i.l. by an appropriate choice of the relay constants c_i and d_i when $B^{\dagger}$ is not an M-matrix.

When $n=2$, various trials made by the author are suggesting a negative answer.

Example 4.2.

Consider a sampled-data composite system

$$x_i(\tau+1)=\sum_{j=1}^{2} g_{ij}(x_j(\tau)) \qquad i=1,2 \qquad (4\text{-}56)$$

where x_1 and x_2 are two dimensional vectors and $g_{ij}(x_j)$ are given by

$$g_{ii}(x_i)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}x_i+\varphi_i(x_i) \qquad i=1,2$$

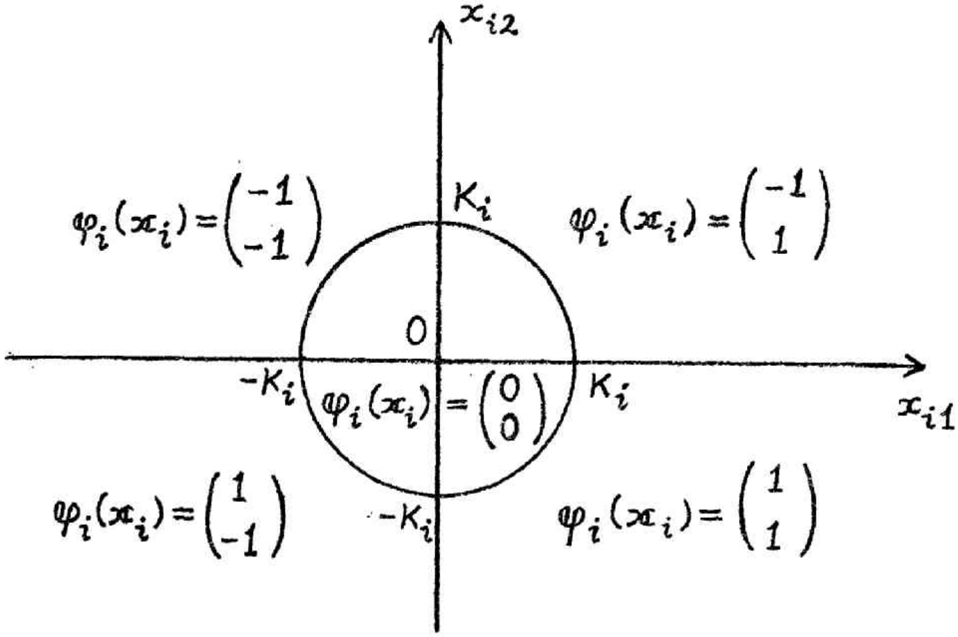
$$g_{ij}(x_j)=\varepsilon_j\,\varphi_j(x_j) \qquad i,j=1,2\ ;\ i\neq j$$

Here $\varphi_i(x_i)$ is a two-dimensional relay function defined in Fig. 4.4. Let Γ_i the disk centered at the origin in which $\varphi_i(x_i)=0$ and let K_i the radius of the disk Γ_i. Let us assume $K_i>1$. Then, we obtain

$$\begin{aligned}\|g_{ii}(x_i)\|-\|x_i\| &\\ &=\left\{(|x_{i1}|-1)^2+(|x_{i2}|-1)^2\right\}^{\frac{1}{2}}-\|x_i\|\\ &\leqq\left\{(\|x_i\|-1)^2+1\right\}^{\frac{1}{2}}-\|x_i\|\\ &\leqq -K_i+\left\{(K_i-1)^2+1\right\}^{\frac{1}{2}}<0\end{aligned}$$

$$x_i\notin\Gamma_i\ ;\ i=1,2$$

Fig. 4.4. The Two-Dimensional Relay Function $\varphi_i(x_i)$



For the interaction terms, we have

$$\|g_{ij}(x_j)\| = \sqrt{2}\,|\varepsilon_j| \qquad x_j \notin \Gamma_j \ ; \ i=1,2 \ ; \ i \neq j$$

Therefore, by choosing

$$N_i(x_i) = \|x_i\|$$

$$M_i(x_i) = 1$$

the assumptions (4-i') and (4-ii) of Theorem 4.6. are satisfied. The matrix A is given by

$$A = \begin{pmatrix} K_1 - \sqrt{(K_1-1)^2+1} & -\sqrt{2}\,|\varepsilon_2| \\ -\sqrt{2}\,|\varepsilon_1| & K_2 - \sqrt{(K_2-1)^2+1} \end{pmatrix}$$

By Theorem 4.6., the system is a.s.i.l. if

$$|\varepsilon_1 \varepsilon_2| < \frac{1}{2}\left\{K_1 - \sqrt{(K_1-1)^2+1}\right\}\left\{K_2 - \sqrt{(K_2-1)^2+1}\right\} \tag{4-57}$$

Chapter 5 Application to a System Composed of Subsystems Containing a Single Nonlinearity

Sec. 5.1. Description of the System

Consider a continuous-time system given by

$$\frac{dx_i}{dt} = A_{ii} x_i + \mathbb{b}_{ii} \varphi_i(\sigma_i) + C_i u_i \tag{5-1}$$

$$\sigma_i = \mathbb{d}_i^t x_i \tag{5-2}$$

$$u_i = \sum_{j;j\neq i} v_{ij} \tag{5-3}$$

$$v_{ij} = A_{ij} x_j + \mathbb{b}_{ij} \varphi_j(\sigma_j) + E_{ij} y_{ij} \qquad i \neq j \tag{5-4}$$

$$\frac{dy_{ij}}{dt} = F_{ij} y_{ij} + G_{ij} x_j + \mathbb{h}_{ij} \varphi_j(\sigma_j) \qquad i \neq j \tag{5-5}$$

Here

x_i : m_i -dimensional vector

u_i : m'_i -dimensional vector

v_{ij} : m'_i -dimensional vector

y_{ij} : m''_i -dimensional vector

A_{ii} : $m_i \times m_i$ constant matrix

$\mathbb{b}_{ii}$: m_i-dimensional constant vector

C_i : $m_i \times m'_i$ constant matrix

$\mathbb{d}$: m_i -dimensional constant vector

A_{ij} $(i \neq j)$: $m'_i \times m_j$ constant matrix

$\mathbb{b}_{ij}$ $(i \neq j)$: m'_i-dimensional constant vector

$E_{ij}\ (i \neq j)$: $m'_i \times m''_{ij}$ constant matrix

$F_{ij}\ (i \neq j)$: $m''_{ij} \times m''_{ij}$ constant matrix

$G_{ij}\ (i \neq j)$: $m''_{ij} \times m_j$ constant matrix

$\mathbb{h}_{ij}\ (i \neq j)$: m''_{ij}-dimensional constant vector

and $\varphi_i(\sigma_i)$'s are continuous functions satisfying

$$\varphi_i(0) = 0 \ , \ 0 \leqq \frac{\varphi_i(\sigma_i)}{\sigma_i} \leqq k_i \quad (\sigma_i \neq 0) \quad (5\text{-}6)$$

We can interpret the above set of equations as follows (cf. Fig. 5.1. and Fig. 5.2.). Eqs. (5-1) and (5-2) describe a system with single nonlinearity φ_i and an input u_i (Fig. 5.1.). Let us call this system the <u>i-th</u> <u>intrinsic</u> <u>subsystem</u> ISS$_i$. Eq. (5-3) tells that the input to the intrinsic subsystem ISS$_i$ is the sum of the interactions v_{ij} from the other intrinsic subsystems (Fig. 5.1.). Eqs. (5-4) and (5-5) describe the property of the interaction v_{ij}. Eq. (5-4) tells that the interaction v_{ij} from ISS$_j$ to ISS$_i$ consists of two parts : the direct interaction given by the term $A_{ij} x_j + \mathbb{b}_{ij} \varphi_j(x_j)$ and the delayed interaction given by the term $E_{ij} y_{ij}$ (Fig. 5.2.). Eq. (5-5) gives the dynamical property of the delayed interaction. Let us call the system described by (5-5) the <u>i-j</u> <u>connecting</u> <u>subsystem</u> CSS$_{ij}$, where we regard the term $G_{ij} x_j + \mathbb{h}_{ij} \varphi_j(x_j)$ as the input. By the word

Fig. 5.1. The i-th Intrinsic Subsystem

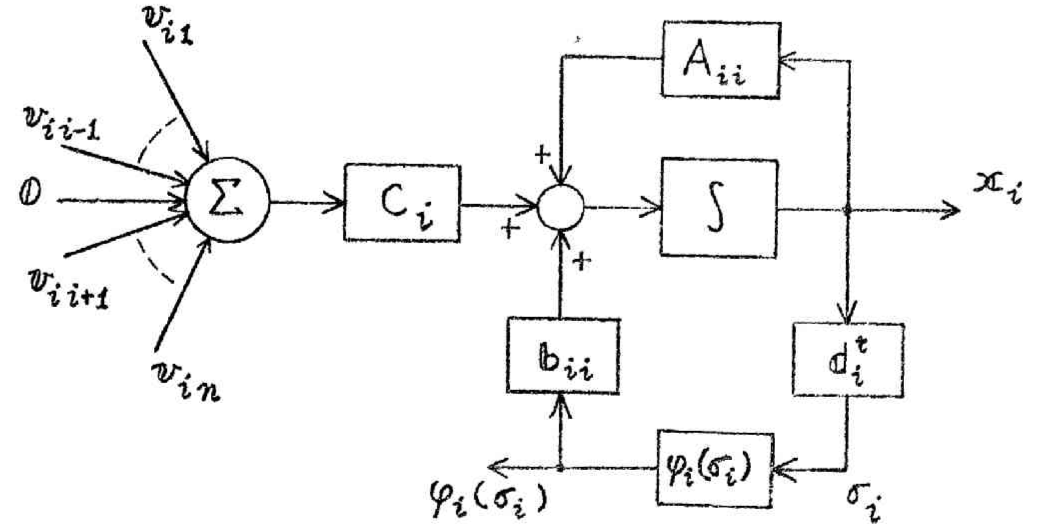
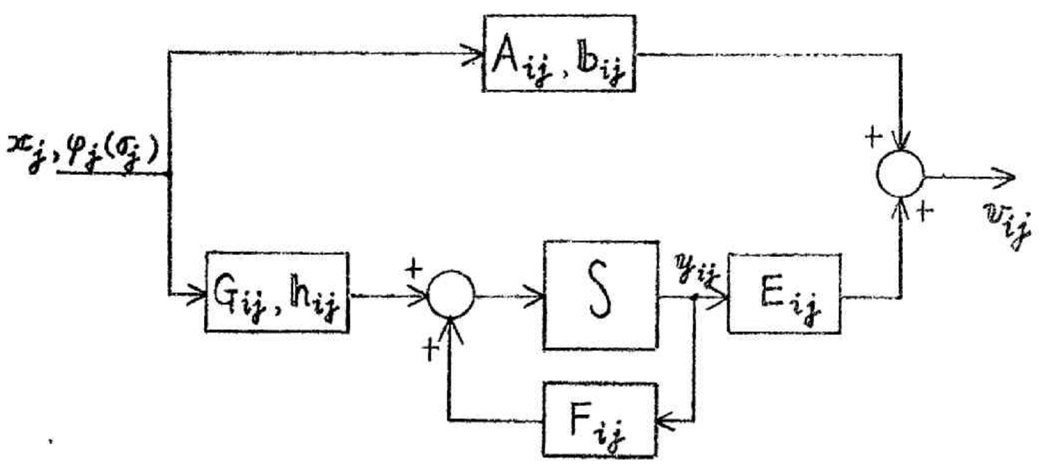


Fig. 5.2. The i-j Connection



subsystem, we refer to the both kinds of subsystems : intrinsic subsystems and connecting subsystems.

The above kind of model is obtained for a multivariable control system when the actuators have noticeable nonlinear properties and the other part of the system can be regarded as having linear property [14, 50]. When we construct the above kind of model for an engineering system, we often encounter the case in which we must replace (5-3) by

$$abs(u_i) \leq \sum_{j;j\neq i} abs(v_{ij}) \tag{5-3-a}$$

(cf. 3.4.1. of Sec. 3.4.). For such a case, the stability analysis given in the following sections are applicable without any change. (cf. Theorem 3.4.) An important example, which belongs to this category, is the case in which $m_i' = 1$ and in which we can construct the equation of an intrinsic subsystem in the form

$$\frac{dx_i}{dt} = A_{ii} x_i + \mathbb{b}_{ii} \varphi_i(\sigma_i)$$

$$\sigma_i = \mathbb{d}_i^t x_i + u_i$$

$$u_i = \sum_{j;j\neq i} v_{ij}$$

In such a case, if we can assume

$$\varphi_i(0)=0 \ , \ 0 \leqq \frac{\varphi_i(\sigma_i)-\varphi_i(\sigma_i')}{\sigma_i-\sigma_i'} \leqq k_i \quad (\sigma_i \neq \sigma_i')$$

(5-6+)

we obtain

$$\frac{dx_i}{dt} = A_{ii}x_i + b_{ii}\varphi_i(\sigma_i) + C_i u_i \qquad (5\text{-}1)$$

$$C_i = k_i b_{ii}$$

$$\sigma_i = d_i^t x_i \qquad (5\text{-}2)$$

$$abs(u_i) \leqq \sum_{j;j \neq i} abs(v_{ij}) \qquad (5\text{-}3\text{-}a)$$

Now, in parallel with the continuous-time case described above, consider a sampled-data system given by

$$x_i(\tau+1) = A_{ii}x_i(\tau) + b_{ii}\varphi_i(\sigma_i(\tau)) + C_i u_i(\tau) \qquad (5\text{-}7)$$

$$\sigma_i(\tau) = d_i^t x_i(\tau) \qquad (5\text{-}8)$$

$$u_i(\tau) = \sum_{j;j \neq i} v_{ij}(\tau) \qquad (5\text{-}9)$$

$$v_{ij}(\tau) = A_{ij}x_j(\tau) + b_{ij}\varphi_j(\sigma_j(\tau)) + E_{ij}y_{ij}(\tau) \qquad i \neq j \qquad (5\text{-}10)$$

$$y_{ij}(\tau+1) = F_{ij}y_{ij}(\tau) + G_{ij}x_j(\tau) + h_{ij}\varphi_j(\sigma_j(\tau)) \qquad i \neq j \qquad (5\text{-}11)$$

Here, x_i , u_i , v_{ij} , y_{ij} , A_{ij} , b_{ij} , C_i , d_i , E_{ij} ,

F_{ij} , G_{ij} , $\mathbb{h}_{ij}$ are as defined in the continuous-time case and $\varphi_i(\sigma_i)$'s are functions satisfying (5-6).

We can interpret the above equations just in the same way as the continuous-time case. If we introduce a digital computer in the feedback loop, we obtain this kind of model. In such a case, both the nonlinear property of the actuator and the nonlinearity of the quantizer may be involved in the function $\varphi_i(\sigma_i)$. Here, in parallel with the continuous-time case, we call the system given by (5-7) and (5-8) the <u>i-th intrinsic subsystem</u> ISS$_i$ and the system given by (5-11) the <u>i-j connecting subsystem</u> CSS$_{ij}$.

<u>Sec</u>. 5.2. General Procedure of Stability Analysis in Continuous-Time Case

In this section, a general procedure of investigating the stability of the continuous-time system given by (5-1) - (5-5) will be given.

5.2.1. Stability Criterion

Concerning a.s.i.l. of the system given by (5-1) - (5-5), we have the next theorem, which is obtained by applying Theorem 3.1. and considering the structure of the system.

Theorem 5.1.

Consider the system given by (5-1) - (5-5).

Assume*

(5-i) that, for each intrinsic subsystem ISS_i , we have a Lyapunov function

$$v_i(x_i) = x_i^t P_{ii} x_i + \theta_i \int_0^{\sigma_i} \varphi_i(\xi)\, d\xi \qquad (5\text{-}12)$$

such that

$$\left(\frac{dv_i}{dt}\right)_{(5\text{-}1),(5\text{-}2),u_i=0} \leq -\begin{pmatrix} x_i \\ \varphi_i \end{pmatrix}^t Q_{ii} \begin{pmatrix} x_i \\ \varphi_i \end{pmatrix} \qquad (5\text{-}13)$$

where P_{ii} is an m_i-th order positive definite matrix and Q_{ii} is an (m_i+1)-th order positive definite matrix,

(5-ii) and that, for each connecting subsystem we have a pair of m''_{ij}-th order positive definite matrixes P_{ij} and Q_{ij} such that

---

* $\begin{pmatrix} x_i \\ \varphi_i \end{pmatrix}$ is the (m_i+1)-dimensional vector, the first m_i components of which are the components of x_i and the last component of which is φ_i .

$$F_{ij}^t P_{ij} + P_{ij} F_{ij} = -Q_{ij} \tag{5-14}$$

The system is a.s.i.l. if

(5-iii) the n-th order matrix $A^{\dagger}=(a^{\dagger}_{ij})$ is an M-matrix, i.e. the leading principal minor determinants of $A^{\dagger}$ are all positive (cf. Definition A.1. in Appendix).

Here, $a^{\dagger}_{ij}$'s are given by

$$a^{\dagger}_{ii} = \frac{\mu_{ii}}{2\lambda_{ii} + k_i |\theta_i| \cdot \|d_i\|^2} \tag{5-15-1}$$

$$a^{\dagger}_{ij} = -\left\{ \|C_i A_{ij}\| + k_j \|C_i b_{ij}\| \cdot \|d_j\| + \frac{2\lambda_{ij}}{\mu_{ij}} \|C_i E_{ij}\| \times \left(\|G_{ij}\| + k_j \|h_{ij}\| \cdot \|d_j\| \right) \right\} \quad i \neq j \tag{5-15-2}$$

where λ_{ii} and λ_{ij} are respectively the maximum characteristic roots of P_{ii} and P_{ij}, and μ_{ii} and μ_{ij} are respectively the minimum characteristic roots of Q_{ii} and Q_{ij}.

Theorem 5.2.

Consider the system given by (5-1) - (5-5). Assume (5-i) and (5-ii). The system is a.s.i.l. if

(5-iii+) the n-th order matrix $A^{*}=(a^{*}_{ij})$ is an M-matrix where a^{*}_{ij}'s are given by

$$a^{*}_{ii} = \frac{\mu_{ii}}{(2\lambda_{ii} + k_i |\theta_i| \cdot \|d_i\|^2) \|C_i\|} \tag{5-15-3}$$

$$a^*_{ij} = -\left\{ \|A_{ij}\| + k_j \|\mathbb{b}_{ij}\| \cdot \|\mathbb{d}_j\| + \frac{2\lambda_{ij}}{\mu_{ij}} \|E_{ij}\| \right.$$

$$\left. \times \left(\|G_{ij}\| + k_j \|\mathbb{h}_{ij}\| \cdot \|\mathbb{d}_j\| \right) \right\} \quad i \neq j \qquad (5\text{-}15\text{-}4)$$

where λ_{ii}, λ_{ij} , μ_{ii} and μ_{ij} are as given in (5-iii) of Theorem 5.1.

The proof of Theorem 5.1. is given in 5.2.2. If we note the relations

$$\|C_i A_{ij}\| \leq \|C_i\| \, \|A_{ij}\| \quad , \quad \|C_i \mathbb{b}_{ij}\| \leq \|C_i\| \, \|\mathbb{b}_{ij}\|$$

$$\|C_i E_{ij}\| \leq \|C_i\| \, \|E_{ij}\|$$

and if we note Theorem A.2. and A.3. in Appendix, Theorem 5.2. is evident by Theorem 5.1. Theorem 5.2. has the advantage that the off-diagonal elements a^*_{ij} $(i \neq j)$ do not contain any constant of the i-th intrinsic subsystem ISS$_i$. When the dimensions m'_i of the connections are all one (<u>scalor</u> <u>connections</u>), (5-iii+) is equivalent to (5-iii), and otherwise (5-iii+) is generally more conservative than (5-iii).

The above theorems show a general procedure to investigate the stability of the system given by (5-1) - (5-5). First, for each intrinsic subsystem, construct a Lyapunov function of the form (5-12). Next, for each connecting subsystem, find out a pair of P_{ij} and Q_{ij} satisfying (5-14). For this purpose, give an arbitrary positive definite matrix Q_{ij} and solve the equation (5-14) for P_{ij}. If F_{ij} is a stable matrix, the solution P_{ij} is always positive definite (cf. [9], p. 245 of [22], etc.). If we cannot obtain the Lyapunov function v_i or the pair of matrixes P_{ij} and Q_{ij} for some subsystems, we cannot use our theorem. If we can fulfil the above steps for all subsystems, we can apply the condition (5-iii).

Concerning the construction of a Lyapunov function of the form (5-12) for the intrinsic subsystem, the literatures [2-6, 38-41] provide various useful results. In order to make the condition (5-iii) less conservative, it is profitable to make a^{+}_{ii} as large as possible. (cf. Theorem A.2. in Appendix) For this purpose, it is desirable to choose the elements of P_{ii} so that we obtain the maximum a^{+}_{ii}. But, generally, it is a difficult work to maximize a^{+}_{ii} except the case in which m_i is small. Apparently, it seems that we obtain larger a^{+}_{ii} if we make $|\theta_i|$ smaller. But this is not true because the matrix Q_{ii} (hence the number μ_i) depends upon θ_i.

The research by Yakubovich [38-41] gives an interesting knowledge concerning the construction of the Lyapunov function for an intrinsic subsystem. For the i-th intrinsic subsystem, put

$$\mathcal{G}_i(s) = \mathbb{d}_i^t (A_{ii} - sI)^{-1} \mathbb{b}_{ii} \tag{5-16}$$

Assume that the characteristic roots of A_{ii} exist only in the half plane

$$\mathrm{Re}(s) < -\xi_i \tag{5-17}$$

where ξ_i is a positive constant. Assume that there is a positive constant θ_i such that*, **

$$\pi(\omega) \equiv k_i^{-1} + Re\{(1+i\omega\theta_i)\mathcal{G}_i(-\xi_i+i\omega)\} > 0$$
$$-\infty < \omega < \infty \tag{5-18}$$

$$\lim_{\omega\to\infty} \pi(\omega)\,\omega^2 > 0 \quad \text{if} \quad \lim_{\omega\to\infty} \pi(\omega) = 0 \tag{5-19}$$

---

* Note that i is used in two meanings : the imaginary unit and the subscript to identify the subsystem.

** The condition given by (5-18) tells that we have a Popov line of the gradient θ_i for the modified frequency response curve $\mathcal{G}_i^*(-\xi_i+i\omega)$, where

$$\mathcal{G}_i^*(-\xi_i+i\omega) = Re(\mathcal{G}_i(-\xi_i+i\omega)) + i\omega Im(\mathcal{G}_i(-\xi_i+i\omega$$

(cf. [39])

Then, there exists a Lyapunov function of the form (5-12) such that*

$$\left(\frac{d v_i}{d t}\right)_{(5-1),(5-2),u_i=0} \leqq -2\xi_i v_i \qquad (5\text{-}20)$$

(cf. Yakubovich [39]). Here, let T_i an m_i-th order non-singular matrix such that

$$T_i^t P_{ii} T_i = I$$

By transforming the state vector of the i-th intrinsic subsystem by

$$z_i = T_i^{-1} x_i$$

we obtain

$$\frac{d z_i}{dt} = T_i^{-1} A_{ii} T_i z_i + T_i^{-1} b_{ii} \varphi_i(\sigma_i) + T_i^{-1} C_i u_i \qquad (5\text{-}1')$$

$$\sigma_i = d_i^t T_i z_i \qquad (5\text{-}2')$$

and

$$v_i(z_i) = \|z_i\|^2 + \theta_i \int_0^{\sigma_i} \varphi_i(\xi)\, d\xi \qquad (5\text{-}12')$$

By using the expression (5-1') and (5-2'), we obtain

$$a_{ii}^{\dagger} = \xi_i K_i(\theta_i, T_i) \qquad (5\text{-}21)$$

where

---

* Eq. (5-20) means the i-th intrinsic subsystem is exponentially stable with the exponent ξ_i (cf. p. 40).

$$K_i(\theta_i, T_i) = \begin{cases} \dfrac{1}{1 + \frac{1}{2} k_i \theta_i \| d_i^t T_i \|^2} & \theta_i \geq 0 \\ \dfrac{1 - \frac{1}{2} k_i |\theta_i| \cdot \| d_i^t T_i \|^2}{1 + \frac{1}{2} k_i |\theta_i| \cdot \| d_i^t T_i \|^2} & \theta_i < 0 \end{cases} \tag{5-22}$$

Here we must note that, in (5-15-2), c_i and d_j must be replaced by $T_i^{-1} c_i$ and $d_j^t T_j$, respectively. The above procedure is theoretically interesting because it relates the constant a^+_{ii} to the knowledge obtained by the graphical method. But it requires complex calculation to obtain the matrix P_{ii} [38] and T_i. In some cases, the matrix T_i becomes nearly non-singular and the norm of $T_i^{-1} c_i$ becomes very large. Therefore, the above procedure using Yakubovich's results is not so recommendable for practical application in spite of its theoretical importance.

Concerning the pair of positive definite matrixes P_{ij} and Q_{ij} for the connecting subsystem, the Kalman's corollary (p. 380 of [9]) gives a method of transforming the state vector parallel with the above method based on the Yakubovich's result. But, either in this case, the method is not so recommendable for practical purpose because of the same reasons.

5.2.2. Proof of Theorem 5.1. and a Supplement

Let us regard the system given by (5-1) - (5-5) as composed of n^2 subsystems, n of which are the intrinsic subsystems and $n(n-1)$ of which are the connecting subsystems. By applying Theorem 3.1., we will obtain, as a stability condition, that an n^2-th order matrix A be an M-matrix. By considering the structure of the system, we can reduce this condition to (5-iii).

Proof of Theorem 5.1. :

Let us number the n^2 subsystems with the integer parameter ℓ, where

$$\ell = n \times (i-1) + i \tag{5-23}$$

for the i-th intrinsic subsystem and

$$\ell = n \times (i-1) + j \qquad i \neq j \tag{5-24}$$

for the i-j connecting subsystem*.

First, let us show that the assumption (3-i) is

---

* In this chapter, ℓ takes the values $1, ---, n^2$. The subscripts i and j take the values $1, ---, n$ as said in the footnote at p. 10.

satisfied. Consider the i-th intrinsic subsystem and put $v_\ell(x_i) = v_i(x_i)$ By (5-12)

$$\nabla_i v_\ell(x_i) = 2P_{ii}x_i + \theta_i d_i \varphi_i(\sigma_i) \qquad (5\text{-}25)$$

By (5-6),

$$|\nabla_i v_\ell(x_i)| \leqq 2\lambda_{ii}\|x_i\| + k_i \theta_i \|d_i\|^2 \|x_i\| \quad (5\text{-}26)$$

Therefore, eq. (3-3) is satisfied where

$$\delta_\ell = 2\lambda_{ii} + k_i|\theta_i|\,\|d_i\|^2 \qquad \ell = n(i-1)+i \qquad (5\text{-}27)$$

By (5-13),

$$\left(\frac{dv_i}{dt}\right)_{(5\text{-}1),(5\text{-}2),u_i=0} \leqq -\mu_{ii}(\|x_i\|^2 + \varphi_i^2) \leqq -\mu_{ii}\|x_i\|^2 \qquad (5\text{-}28)$$

Therefore, eq. (3-2) is satisfied where

$$\gamma_\ell = \mu_{ii} \qquad \ell = n(i-1)+i \qquad (5\text{-}29)$$

Next, consider the i-j connecting subsystem. By putting

$$v_\ell(x_\ell) = y_{ij}^t P_{ij} y_{ij} \quad \ell = n(i-1)+j\,;\ i \neq j \qquad (5\text{-}30)$$

we obtain

$$\left(\frac{dv_\ell}{dt}\right)_{(5\text{-}5),x_j=0} = -y_{ij}^t Q_{ij} y_{ij} \qquad (5\text{-}31)$$

$$\ell = n(i-1)+j\,,\ i \neq j$$

Therefore, eqs. (5-2) and (5-3) are satisfied where

$$\gamma_\ell = \mu_{ij} \;,\; \delta_\ell = 2\lambda_{ij} \qquad \ell = n(i-1)+j \;,\; i \neq \tag{5-32}$$

Eq. (5-1) is satisfied by the assumptions made on v_i and P_{ij} in (5-i) and (5-ii). Therefore assumption (3-i) is satisfied.

Next, let us show that the assumption (3-ii) is satisfied. The equations (5-1) - (5-5) can be rewritten as

$$\begin{aligned}\frac{dx_i}{dt} = A_{ii}\,x_i + \mathbb{b}_{ii}\,\varphi_i(\mathbb{d}_i^{\,t} x_i) \\ + \sum_{j;\,j\neq i}\{C_i A_{ij}\,x_j + C_i\,\mathbb{b}_{ij}\,\varphi_j(\mathbb{d}_j^{\,t} x_j)\} \\ + \sum_{j;\,j\neq i} C_i E_{ij}\,y_{ij}\end{aligned} \tag{5-33}$$

$$\frac{dy_{ij}}{dt} = F_{ij}\,y_{ij} + G_{ij}\,x_j + \mathbb{h}_{ij}\,\varphi_j(\mathbb{d}_j^{\,t} x_j) \tag{5-34}$$

Therefore, (3-ii) is satisfied where

$$\varepsilon_{\ell\ell'} = \|C_i A_{ij}\| + \|C_i\,\mathbb{b}_{ij}\|\;k_j\,\|\mathbb{d}_j\| \equiv \chi_{ij}$$
$$\ell = n(i-1)+i \,,\; \ell' = n(j-1)+j \,,\; i \neq j \tag{5-35-1}$$

$$\varepsilon_{\ell\ell'} = \|C_i E_{ij}\| \equiv \chi'_{ij}$$
$$\ell = n(i-1)+i ,\; \ell' = n(i-1)+j \,,\; i \neq j \tag{5-35-2}$$

$$\varepsilon_{\ell\ell'} = \|G_{ij}\| + \|h_{ij}\| k_j \|d_j\| \equiv \chi''_{ij}$$

$$\ell = n(i-1)+j \ , \ \ell' = n(j-1)+j \ , \ i \neq j \qquad (5\text{-}35\text{-}3)$$

$\varepsilon_{\ell\ell'} = 0$ for the other pairs of ℓ and ℓ' , $\ell \neq \ell'$

(5-35-4)

Therefore, by Theorem 3.1., the system is a.s.i.l. if

(5-iv) the matrix A is an M-matrix, where $A = (a_{\ell\ell'})$ is given by

$$a_{\ell\ell} = \frac{\mu_{ii}}{2\lambda_{ii} + k_i |\theta_i| \cdot \|d_i\|^2} \qquad \ell = n(i-1)+i$$

(5-36-1)

$$a_{\ell\ell} = \frac{\mu_{ij}}{2\lambda_{ij}} \qquad \ell = n(i-1)+j \ , \ i \neq j$$

(5-36-2)

$$a_{\ell\ell'} = -\varepsilon_{\ell\ell'} \qquad \ell \neq \ell' \qquad (5\text{-}36\text{-}3)$$

In the following, let us show that (5-iii) is equivalent to (5-iv).

Let us write the matrix A noticing the ℓ_1-th and ℓ_3-th rows and ℓ_2-th and ℓ_3-th columns (Fig. 5.3.), where

$$\ell_1 = n(i-1)+i \quad , \quad \ell_2 = n(j-1)+j$$

$$\ell_3 = n(i-1)+j \qquad (i \neq j)$$

On the ℓ_3-th column, the elements are all zero except the elements on the ℓ_1-th and ℓ_3-th row. On the ℓ_3-th row, the elements are all zero except the elements on the ℓ_2-th and ℓ_3-th column. In addition we know

$$a_{\ell_1 \ell_2} = -\chi_{ij} \quad , \quad a_{\ell_1 \ell_3} = -\chi'_{ij}$$

$$a_{\ell_3 \ell_2} = -\chi''_{ij} \quad , \quad a_{\ell_3 \ell_3} = \frac{\mu_{ij}}{2\lambda_{ij}}$$

Therefore, the k-th order leading principal minor determinant D_k of A is calculated by adding $(2\lambda_{ij}/\mu_{ij})\chi'_{ij}$ times ℓ_3-th row to the ℓ_1-th row. The resulting matrix is given in Fig. 5.4. In this way, any leading principal minor determinant D_k of A is related to $D^{\dagger}_k \equiv |a^{\dagger}_{ij}|$ $(i,j=1,\cdots,k$ by

$$D_k = D^{\dagger}_{k'} \prod_{(i,j)\in \Psi_k} \frac{\mu_{ij}}{2\lambda_{ij}} \tag{5-37}$$

$$\Psi_k = \{ (i,j) \, ; \, i \neq j \, , \, n(i-1)+j \leq k \}$$

where k' is the maximum integer such that

$$n(k'-1)+k' \leq k \tag{5-38}$$

and $a^{\dagger}_{ij}$ are associated with $a_{\ell\ell'}$ by

Fig. 5.3. The Matrix $A=(a_{\ell\ell'})$

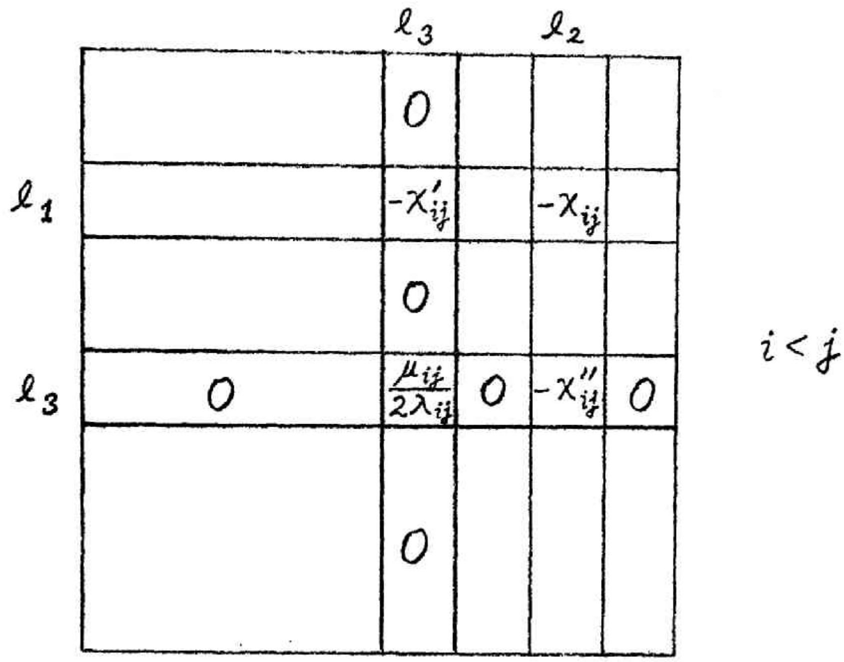
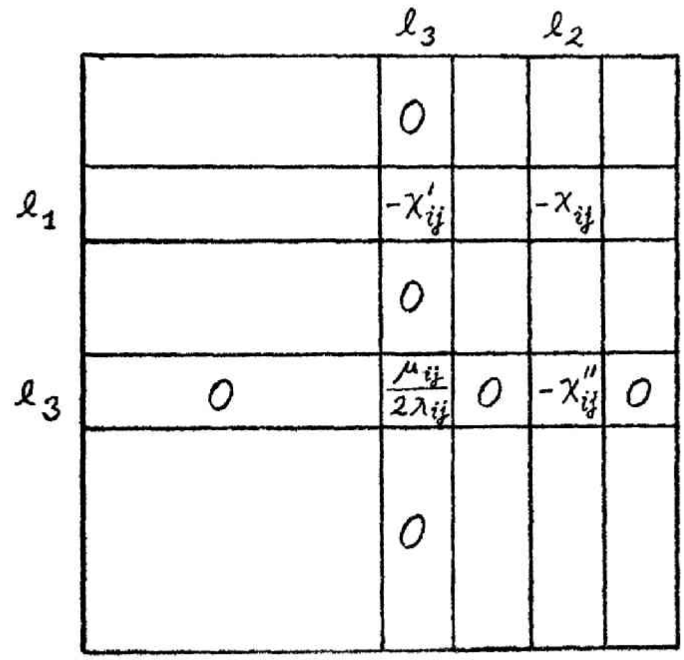


Fig. 5.4. Reduction of the Matrix $A=(a_{\ell\ell'})$



$$a^{+}_{ij} = a_{\ell_1 \ell_1} \tag{5-39-1}$$

$$a^{+}_{ij} = a_{\ell_1 \ell_2} - \frac{a_{\ell_1 \ell_3}\, a_{\ell_3 \ell_2}}{a_{\ell_3 \ell_3}} \tag{5-39-2}$$

By (5-37) and (5-39) and by the definition of an M-matrix, we can say that D_k is positive if A^{+} is an M-matrix. Therefore, we can conclude (5-iv) from (5-iii). The inverse assertion that (5-iii) follows (5-iv) is evident by the fact that any principal minor determinants of A^{+} is equal to a principal minor determinant of A divided by the positive factors $\mu_{ij}/2\lambda_{ij}$, which is shown just in the same way as inducing (5-37).

(Q.E.D.)

Here, corresponding to Theorem 3.2. given in Sec. 3.2., we have the next supplement to Theorem 5.1. and 5.2.

<u>Supplement to Theorem 5.1. and 5.2.</u>

If the elements of the $k^{(i)}_{1}$ -th,---, $k^{(i)}_{p_i}$-th row of the matrix C_i are all zero, we can replace λ_{ii} and $\|d_i\|$ in (5-15-1) and (5-15-3) by λ'_{ii} and $\|\text{modified}(d_i)\|$ where

$$\lambda'_{ii} = \| \text{modified} (P_{ii}) \|$$

If the elements of the $k_1^{(ij)}$-th, ---, $k_{p_{ij}}^{(ij)}$-th row of the matrix G_{ij} and the $k_1^{(ij)}$-th, ---, $k_{p_{ij}}^{(ij)}$-th components of the vector h_{ij} are all zero, we can replace λ_{ij} in (5-15-2) and (5-15-4) by λ'_{ij} where

$$\lambda'_{ij} = \| \text{modified} (P_{ij}) \|$$

(as for modified( ), cf. Theorem 3.2.).

Generally, we can replace μ_{ii} in (5-15-1) and (5-15-3) by a positive number μ'_{ii} such that

$$\begin{pmatrix} x_i \\ \varphi \end{pmatrix}^t Q_{ii} \begin{pmatrix} x_i \\ \varphi \end{pmatrix} \geqq \mu'_{ii} \| x_i \|^2 \tag{5-40}$$

for arbitrary real values of x_{ik} ($k = 1, ---, m_i$) and φ.*

Sec. 5.3. A Continuous-Time System Composed of First and Second Order Subsystems

5.3.1. Description of the System

Here, let us study a composite system given by Fig. 5.5. (at p. 111), where we assume the functions $\varphi_i(\sigma_i)$'s satisfy the relation (5-6).

---

* In (5-40), φ must be regarded as an independent variable.

This system is a special case of the system given by (5-1) - (5-5) (to be accurate, (5-3) must be replaced by (5-3-a)), where $m'_i = 1$ for $i = 1, \cdots, n$. Here, the quantities $\mathbb{A}_{ij}$, $\mathbb{b}_{ij}$, $\mathbb{G}_{ij}$ and $\mathbb{h}_{ij}$ degenerate into scalors. In this section, we confine our study to the cases in which $\mathcal{G}_{ij}$ has one of the next forms :

$$\mathcal{G}_{ij}(s) = \frac{r_{ij}}{1 + T_{ij} s} \tag{5-41}$$

$$\mathcal{G}_{ij}(s) = \frac{1}{1 + 2 \zeta_{ij} T_{ij} s + (T_{ij} s)^2} \tag{5-42}$$

Since the connections are all scalor connections in this system, we can use Theorem 5.2. without any loss of the sharpness of the stability condition. (cf. p. 89) In the following, let us calculate the elements of the matrix A^* of the condition (5-iii+) of Theorem 5.2. and express the stability criterion with the constants T_{ij}, ζ_{ij}, r_{ij}, A_{ij}, b_{ij}, G_{ij}, h_{ij} and k_i.

5.3.2. First Order Intrinsic Subsystem

Let us calculate the diagonal element a^*_{ii} of the matrix A^* of the condition (5-iii+) for the case in which the transfer function $\mathcal{G}_{ii}(s)$ of the i-th intrinsic subsystem is given by (5-41)

$$\mathcal{G}_{ii}(s) = \frac{r_{ii}}{1 + T_{ii} s}$$

where we suppose

$$r_{ii} > 0 \quad , \quad T_{ii} > 0$$

The equation describing the i-th intrinsic subsystem becomes

$$\frac{dx_i}{dt} = -\frac{1}{T_{ii}} x_i + \frac{r_{ii}}{T_{ii}} \varphi_i(-x_i) + \frac{r_{ii}}{T_{ii}} u_i \tag{5-43}$$

Therefore

$$\|C_i\| = \frac{r_{ii}}{T_{ii}}$$

By putting

$$v_i(x_i) = x_i^{\,2}$$

we obtain

$$\left(\frac{dv_i}{dt}\right)_{(5\text{-}43),\, u_i=0} = 2x_i\left(-\frac{1}{T_{ii}} x_i + \frac{r_{ii}}{T_{ii}} \varphi_i(-x_i)\right)$$

$$\leqq -\frac{2}{T_{ii}} x_i^2 \tag{5-44}$$

$$\frac{\partial v_i}{\partial x_i} = 2x_i \tag{5-45}$$

Therefore, we obtain

$$a_{ii}^{*} = \frac{\dfrac{2}{T_{ii}}}{2 \times \dfrac{r_{ii}}{T_{ii}}} = \frac{1}{r_{ii}} \tag{5-46}$$

5.3.3. Second Order Intrinsic Subsystem

Let us calculate the diagonal element a^*_{ii} of A^* in the condition (5-iii+) for the case in which the transfer function $\mathcal{G}_{ii}$ of the i-th intrinsic subsystem is given by (5-42)

$$\mathcal{G}_{ii}(s) = \frac{r_{ii}}{1+2\xi_{ii}T_{ii}s+(T_{ii}s)^2}$$

where we suppose

$$r_{ii}>0\ ,\ \ \xi_{ii}>0\ ,\ \ T_{ii}>0$$

The equation describing the i-th intrinsic subsystem becomes

$$T_{ii}^2\frac{d^2x_i}{dt^2}+2\xi_{ii}T_{ii}\frac{dx_i}{dt}+x_i$$

$$= r_{ii}\varphi_i(-x_i)+r_{ii}u_i \qquad (5\text{-}47)$$

Here, by putting

$$z_1 = x_i\ \ ,\ \ z_2 = T_i\frac{dx_i}{dt} \qquad (5\text{-}48)$$

the equation of the system is expressed as*

---

* From here to eq. (5-73), we ommit the subscript i in order to make the expression simpler.

$$\left.\begin{aligned} \frac{dz_1}{dt} &= \frac{1}{T} z_2 \\ \frac{dz_2}{dt} &= -\frac{1}{T} z_1 - \frac{2\xi}{T} z_2 + \frac{r}{T}\varphi(\sigma) + \frac{r}{T} u \end{aligned}\right\} \quad (5\text{-}49)$$

$$\sigma = -z_1 \quad (5\text{-}50)$$

In the matrix-vector expression used in Sec. 5.1. and 5.2., (5-49) and (5-50) is expressed as

$$\frac{d\mathbf{z}}{dt} = A\mathbf{z} + \mathbf{b}\varphi(\sigma) + Cu \ , \quad \sigma = \mathbf{d}^t \mathbf{z} \quad (5\text{-}51)$$

$$\left.\begin{aligned} A &= \begin{pmatrix} 0 & \frac{1}{T} \\ -\frac{1}{T} & -\frac{2\xi}{T} \end{pmatrix} , \quad \mathbf{b} = \begin{pmatrix} 0 \\ \frac{r}{T} \end{pmatrix} \\ C &= \begin{pmatrix} 0 \\ \frac{r}{T} \end{pmatrix} , \quad \mathbf{d} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{aligned}\right\} \quad (5\text{-}52)$$

Here, put

$$v(\mathbf{z}) = \mathbf{z}^t \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix} \mathbf{z} + \theta \int_0^\sigma \varphi(\xi)\, d\xi \quad (5\text{-}53)$$

where we suppose

$$p_1 > 0 \ , \ p_2 > 0 \ , \ p_1 p_2 - p_3^2 > 0 \ , \ \theta > 0 \quad (5\text{-}54)$$

Since C has the special form as given by (5-52), we can use the first part of the Supplement to Theorem 5.1. and 5.2. (at p. 100), and we obtain

$$\lambda' \equiv \| \text{modified}(\mathrm{P}) \|$$

$$= \left\| \begin{pmatrix} 0 & 0 \\ p_3 & p_2 \end{pmatrix} \right\| = \sqrt{p_2^{\,2} + p_3^{\,2}} \tag{5-55}$$

$$\text{modified}(\mathrm{d}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5-56}$$

Therefore, the denominator of (5-15-3) is given by

$$2\sqrt{p_2^{\,2} + p_3^{\,2}} \;\; \frac{r}{T} \tag{5-57}$$

Now, by differentiating $v(z)$ along the solution of (5-51) for $u = 0$, we obtain (cf. p. 20 of [3])

$$\left(\frac{dv}{dt}\right)_{(5\text{-}51),\,u=0} = -S(z, \varphi(\sigma)) - \nu\left(\sigma - \frac{\varphi(\sigma)}{k}\right)\varphi(\sigma)$$

$$\leqq -S(z, \varphi(\sigma)) \tag{5-58}$$

where

$$S(z, \varphi) \equiv \begin{pmatrix} z \\ \varphi \end{pmatrix}^{t} Q \begin{pmatrix} z \\ \varphi \end{pmatrix} \tag{5-59}$$

$$Q = Q^{t} = (q_{kk'}) \qquad k, k' = 1, 2, 3$$

$$= \left(\begin{array}{c|c|c} \frac{2p_3}{T} & -\frac{p_1}{T} + \frac{p_2}{T} + \frac{2\xi p_3}{T} & -\frac{rp_3}{T} + \frac{\nu}{2} \\ \hline & \frac{4\xi p_2}{T} - \frac{2p_3}{T} & -\frac{rp_2}{T} + \frac{\theta}{2T} \\ \hline & & \frac{\nu}{k} \end{array}\right) \tag{5-60}$$

$$\nu > 0 \tag{5-61}$$

Here, let us search the positive number μ which satisfies (cf. Supplement to Theorem 5.1. and 5.2. at p. 100)

$$S(\mathbf{z}, \varphi) \geqq \mu' \| \mathbf{z} \|^2 \tag{5-40}$$

(note that, in (5-40), $\mathbf{z}$ and φ are not related by (5-51) but are independent.) Here, we can vary p_1 , p_2 , p_3 , ν and θ arbitrarily under the constraints (5-54) and (5-61). The number μ' cannot be greater than q_{11} nor q_{22} , and equal to the minimum of q_{11} and q_{22} when the off-diagonal elements of Q are all zero. So, let us confine our search to the region where the off-diagonal elements of Q are all zero. By putting the off-diagonal elements of Q zero, we obtain

$$\nu = \frac{2r}{T} p_3 \tag{5-62}$$

$$\theta = 2 r p_2 \tag{5-63}$$

$$p_1 = p_2 + 2 \xi p_3 \tag{5-64}$$

By (5-54) and (5-61), we have the constraints

$$\left.\begin{aligned} & p_2 > 0 \; , \; p_3 \geqq 0 \\ & p_2^2 + 2\xi p_2 p_3 - p_3^2 > 0 \end{aligned}\right\} \tag{5-65}$$

The number μ' is given by

$$\mu' = \min\left(\frac{2}{T} p_3 \, , \, \frac{4\zeta}{T} p_2 - \frac{2}{T} p_3\right) \tag{5-66}$$

To obtain $\mu' > 0$, we must keep

$$p_3 > 0 \, , \quad p_2 > \frac{1}{2\zeta} p_3 \tag{5-67}$$

By (5-57) and (5-66), we obtain

$$a^*_{ii} = \frac{\min(p_3 \, , \, 2\zeta p_2 - p_3)}{\sqrt{p_2^2 + p_3^2}} \, \frac{1}{r} \tag{5-68}$$

Here, let us maximize a^*_{ii} under the constraints (5-65) and (5-67). Putting $p_2 = \xi p_3$, we obtain

$$a^*_{ii} = \frac{1}{r} \, F(\xi) \tag{5-69}$$

$$F(\xi) = \min\left(\frac{1}{\sqrt{1+\xi^2}} \, , \, \frac{2\zeta\xi - 1}{\sqrt{1+\xi^2}}\right) \tag{5-70}$$

$$\left.\begin{aligned} &\xi > \frac{1}{2\zeta} \\ &\xi \geqq -\zeta + \sqrt{\zeta^2 + 1} \end{aligned}\right\} \tag{5-71}$$

In the interval of (5-71), $1/\sqrt{1+\xi^2}$ is monotonic decreasing and $(2\zeta\xi - 1)/\sqrt{1+\xi^2}$ is monotonic increasing. Therefore, the maximum of $F(\xi)$ is given by the value of ξ satisfying

$$\frac{1}{\sqrt{1+\xi^2}} = \frac{2\zeta\xi - 1}{\sqrt{1+\xi^2}} \tag{5-72}$$

Therefore, we obtain the maximum of a^*_{ii} as

$$a^*_{ii} = \frac{1}{\lambda_{ii}} \frac{1}{\sqrt{1+\left(\frac{1}{\zeta_{ii}}\right)^2}} \tag{5-73}$$

5.3.4. Connection between Intrinsic Subsystems

Let us calculate the off-diagonal element a^*_{ij} of the matrix A^* of the condition (5-iii+). In the case of the system given by Fig. 5.5., the factors of (5-15-4) are given by

$$\|A_{ij}\| = |A_{ij}| \ , \ \|b_{ij}\| = |b_{ij}|$$

$$\|d_j\| = 1 \ , \ \|E_{ij}\| = 1$$

Here, we are required to calculate the number

$$\frac{2\lambda_{ij}}{\mu_{ij}} \left(\|G_{ij}\| + k_j \|h_{ij}\| \right)$$

When the transfer function $g_{ij}(s)$ of the connecting subsystem is given by (5-41)

$$g_{ij}(s) = \frac{\lambda_{ij}}{1+T_{ij}s} \quad , \ T_{ij} > 0$$

we obtain the equation of the i-j connecting subsystem as

$$\frac{dy_{ij}}{dt} = -\frac{1}{T_{ij}} y_{ij} + \frac{r_{ij}}{T_{ij}} \left(G_{ij} x_j + h_{ij} \varphi_j(\sigma_j) \right)$$

Hence, we obtain

$$\| G_{ij} \| = \frac{|r_{ij}|}{T_{ij}} |G_{ij}| \ , \quad \| h_{ij} \| = \frac{|r_{ij}|}{T_{ij}} |h_{ij}|$$

and just in the same way as the case of the intrinsic subsystem given in 5.3.2., we obtain

$$\frac{\mu_{ij}}{2\lambda_{ij}} = \frac{1}{T_{ij}}$$

Therefore, we obtain

$$a^*_{ij} = |A_{ij}| + k_j |b_{ij}| + |r_{ij}| \left(|G_{ij}| + k_j |h_{ij}| \right) \tag{5-74}$$

Likewise, when the transfer function is given by (5-42)

$$g_{ij}(s) = \frac{r_{ij}}{1 + 2\zeta_{ij} T_{ij} s + (T_{ij} s)^2} \ , \quad \zeta_{ij} > 0, \ T_{ij} > 0$$

we obtain

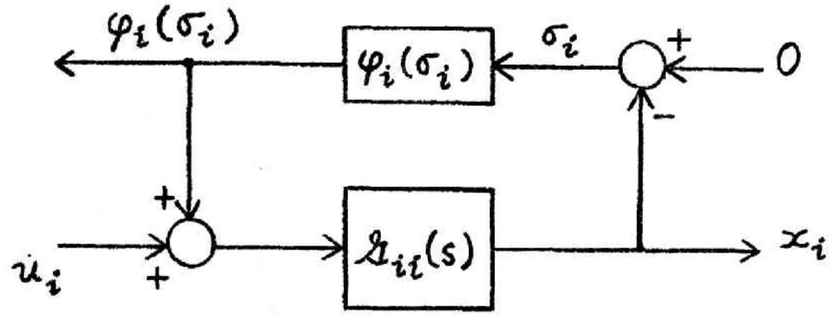
$$a^*_{ij} = |A_{ij}| + k_j |b_{ij}| + |r_{ij}| \sqrt{1 + \left(\frac{1}{\zeta_{ij}}\right)^2} \left(|G_{ij}| + k_j |h_{ij}| \right) \tag{5-75}$$

Fig. 5.5. A Continuous-Time System Composed of

n Intrinsic Subsystems

$(i, j = 1, \cdots, n)$

Fig. 5.5.a. The i-th Intrinsic Subsystem



$$\varphi_i(0) = 0 \ , \ 0 \leqq \frac{\varphi_i(\sigma_i)}{\sigma_i} \leqq k_i \quad (\sigma_i \neq 0) \qquad (5\text{-}6)$$

$$|u_i| \leqq \sum_{j=1, j \neq i}^{n} |v_{ij}| \qquad (5\text{-}3\text{-}a)$$

Fig. 5.5.b. The Interaction from the j-th Intrinsic Subsystem to the i-th Intrinsic Subsystem

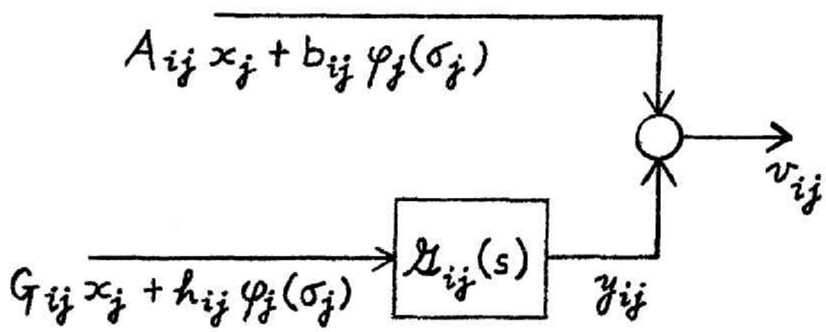


Table 5.1. Elements of the Matrix A^* for the System Given in Fig. 5.5.

Table 5.1.a. Diagonal Elements

$\mathcal{G}_{ii}(s)$	a^*_{ii}
$\dfrac{r_{ii}}{1+T_{ii}s}$	$\dfrac{1}{r_{ii}}$
$\dfrac{r_{ii}}{1+2\zeta_{ii}T_{ii}s+(T_{ii}s)^2}$	$\dfrac{1}{r_{ii}\sqrt{1+\left(\frac{1}{\zeta_{ii}}\right)^2}}$

$$r_{ii}>0\ ,\ T_{ii}>0\ ,\ \zeta_{ii}>0$$

Table 5.1.b. Off-diagonal Elements

$\mathcal{G}_{ij}(s)$	a^*_{ij}
$\dfrac{r_{ij}}{1+T_{ij}s}$	$-(\lvert A_{ij}\rvert+k_j\lvert b_{ij}\rvert)$ $-\lvert r_{ij}\rvert(\lvert G_{ij}\rvert+k_j\lvert h_{ij}\rvert)$
$\dfrac{r_{ij}}{1+2\zeta_{ij}T_{ij}s+(T_{ij}s)^2}$	$-(\lvert A_{ij}\rvert+k_j\lvert b_{ij}\rvert)$ $-\lvert r_{ij}\rvert\sqrt{1+\left(\frac{1}{\zeta_{ij}}\right)^2}\,(\lvert G_{ij}\rvert+k_j\lvert h_{ij}\rvert)$

$$T_{ij}>0\ ,\ \zeta_{ij}>0\ ,\ k_j\geq 0$$

5.3.5. Stability Criterion

<u>Theorem</u> 5.3. (Stability Criterion of the System Given in Fig. 5.5.)

The system given in Fig. 5.5. is a.s.i.l. if the matrix $A^*=(a^*_{ij})$ is an M-matrix, i.e.

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \qquad k = 1, \cdots, n$$

Here, a^*_{ij} are given in Table 5.1. (This theorem has a supplement at p. 137)

In this section, we calculated the elements a^*_{ij} of the matrix A^* only for the first and second order transfer functions given by (5-41) and (5-42). It is not so difficult to obtain a haphazard value of a^*_{ij} for transfer functions of order 3 or 4, but it is difficult to maximize them.

<u>Sec</u>. 5.4. Examples of Continuous-Time System

Several examples are given in order to show the usage of the results obtained in the last section and in order to study the sharpness of the condition.

Example 5.1.

Consider the system given in Fig. 5.6. where $\varphi_i(x_i)$'s are functions satisfying (5-6) and $f_i(x_i)$'s are functions satisfying

$$f_i(0)=0\,, \quad \left|\frac{f_i(x_i)}{x_i}\right| \leq \varepsilon_i \quad (x_i \neq 0) \qquad (5\text{-}76)$$

If we replace $f_i(x_i)$'s with linear gains, this system becomes a special case of Example 3.1.

Let us apply Theorem 5.3. We have

$$G_{ii}(s) = G_i(s)\,; \; G_{ij}(s) = 0\,, \quad b_{ij} = 0 \quad (i \neq j)$$

$$A_{ij} = \begin{cases} \varepsilon_i & i=j+1\,,\; j=1,\cdots,n-1 \\ \varepsilon_n & i=1\,,\; j=n \\ 0 & \text{for the other pairs of } i \text{ and } j\,,\; i\neq j \end{cases}$$

Therefore we obtain the stability condition

$$\prod_{i=1}^{n} r_i^* \times \varepsilon_i < 1 \qquad (5\text{-}77)$$

where

$$r_i^* = \begin{cases} r_i & G_i(s) = \dfrac{r_i}{1+T_i s} \\ r_i\sqrt{1+\dfrac{1}{\zeta_i^2}} & G_i(s) = \dfrac{r_i}{1+2\zeta_i T_i s + (T_i s)^2} \end{cases}$$

As a special case of the above system, let us consider the system given in Fig. 5.7. In this case the stability condition becomes

Fig. 5.6. The Block Diagram of Example 5.1.

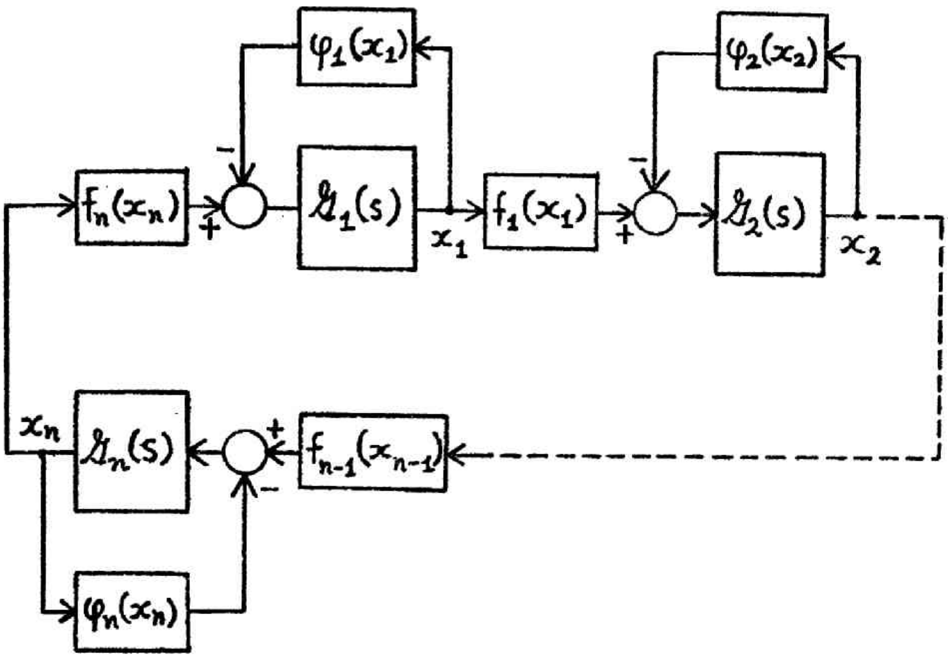
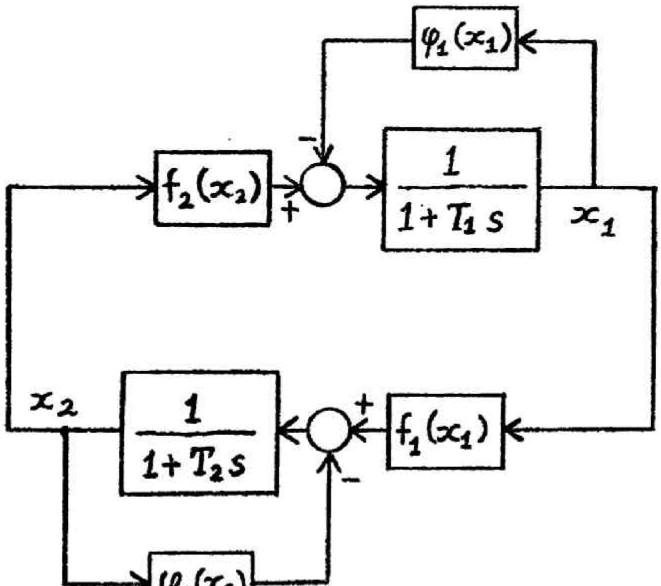


Fig. 5.7. A Special Case of Fig. 5.6.



$$\varepsilon_1 \varepsilon_2 < 1 \tag{5-78}$$

This system was studied by Tokumaru & Saito [8] for

$$f_i(0) = 0 \ , \ 0 \leqq \frac{f_i(x_i)}{x_i} \leqq \varepsilon_i \qquad i = 1, 2$$

$$\varphi_i(x_i) \equiv 0 \qquad i = 1, 2$$

and they obtained the same stability condition (5-78).

In the system of Fig. 5.7., let us replace the nonlinear functions $f_i(x_i)$ and $\varphi_i(x_i)$ by linear gains $f'_i x_i$ and $\varphi'_i \sigma_i$ respectively where f'_i and φ'_i are constants. Then, the whole system becomes a linear, time-invariant system. A necessary and sufficient condition for a.s.i.l. of the linearized system is

$$f'_1 f'_2 < (1 + \varphi'_1)(1 + \varphi'_2) \tag{5-79}$$

Therefore, if (5-78) holds, the linearized system is a.s.i.l. for all f'_i and φ'_i such that

$$f'_i \leqq \varepsilon_i \ , \ \varphi'_i \geqq 0 \qquad i = 1; 2 \tag{5-80}$$

Comparing (5-76) and (5-80), where we regard the both equations as requirements on the d-c gain of the connection, we see that the region

$$f'_1 \leqq -\varepsilon_1 \ , \ f'_2 \leqq -\varepsilon_2$$

is excluded in the nonlinear case (5-76). Main reason of this difference is the fact that we only consider the absolute value of the gain of interconnection and neglect the sign of the connection.

Example 5.2.

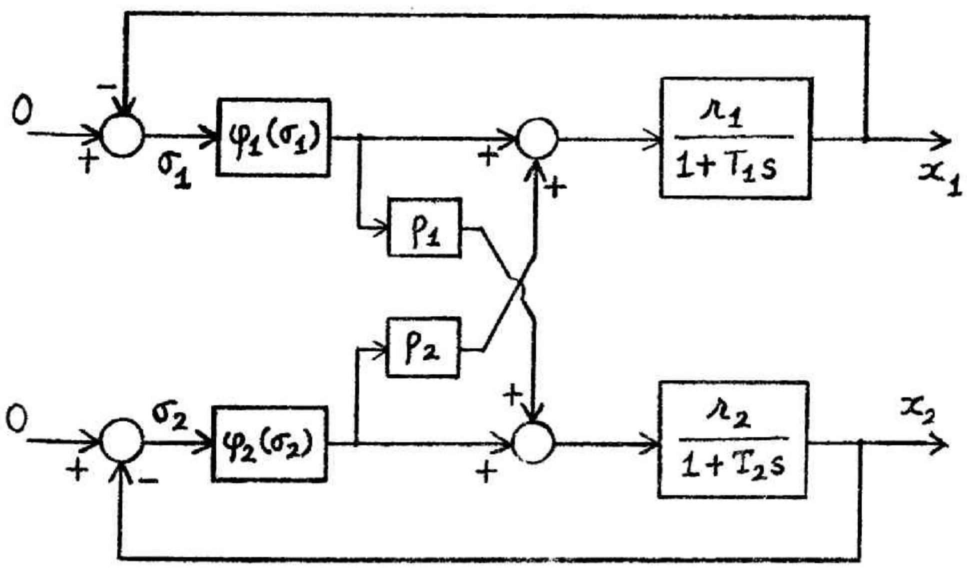
Consider the system given in Fig. 5.8. where $\varphi_1(\sigma_1)$ and $\varphi_2(\sigma_2)$ are functions satisfying (5-6). By applying Theorem 5.3., we obtain the stability condition

$$|\rho_1 \rho_2| < \frac{1}{r_1 r_2 k_1 k_2} \tag{5-81}$$

Here, let us replace $\varphi_1(\sigma_1)$ and $\varphi_2(\sigma_2)$ by linear gains $\varphi_1' \sigma_1$ and $\varphi_2' \sigma_2$ respectively where φ_1' and φ_2' are constants. The characteristic equation of the linearized system is

Fig. 5.8. The Block Diagram of Example 5.2.

$r_1 > 0$, $T_1 > 0$, $r_2 > 0$, $T_2 > 0$



$$\begin{vmatrix} T_1 s + r_1 \varphi_1' + 1 & r_1 \varphi_2' \rho_2 \\ r_2 \varphi_1' \rho_1 & T_2 s + r_2 \varphi_2' + 1 \end{vmatrix} = 0 \quad (5\text{-}82)$$

A necessary and sufficient condition that the linearized system is a.s.i.l. for all

$$0 \leqq \varphi_1' \leqq k_1 \ , \quad 0 \leqq \varphi_2' \leqq k_2$$

is that

$$\rho_1 \rho_2 \leqq \left(1 + \frac{1}{r_1 k_1}\right)\left(1 + \frac{1}{r_2 k_2}\right) \quad (5\text{-}83)$$

Let us compare (5-81) and (5-83), where we regard both equations as requirements on the interconnection constants ρ_1 and ρ_2 . When $r_1 k_1 \ll 1$ and $r_2 k_2 \ll 1$ (weak feedback case), the difference between (5-81) and (5-83) is small except for the negative values of $\rho_1 \rho_2$, but otherwise the difference is pretty great. This example will be studied again in 5.7.1. (at p. 140), where we obtain a sharper stability condition.

<u>Example</u> 5.3.

Consider the system given in Fig. 5.9. where $\varphi_1(\sigma_1)$ and $\varphi_2(\sigma_2)$ are functions satisfying

$$\varphi_i(0) = 0 \ , \ 0 \leqq \frac{\varphi_i(\sigma_i) - \varphi_i(\sigma_i')}{\sigma_i - \sigma_i'} \leqq k_i \ \ (\sigma_i \neq \sigma_i')$$
$$i = 1, 2 \quad (5\text{-}6\ \ )$$

Here, let us convert the system to the form of Fig. 5.5. We have the relation

Fig. 5.9. The Block Diagram of Example 5.3.

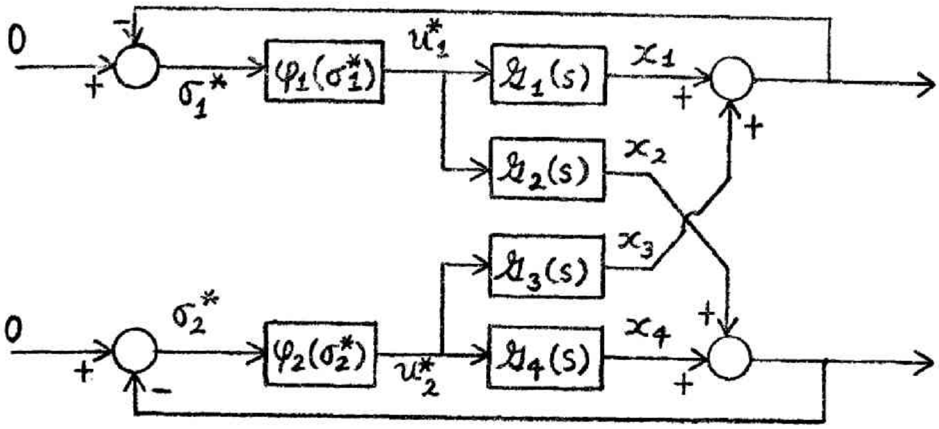
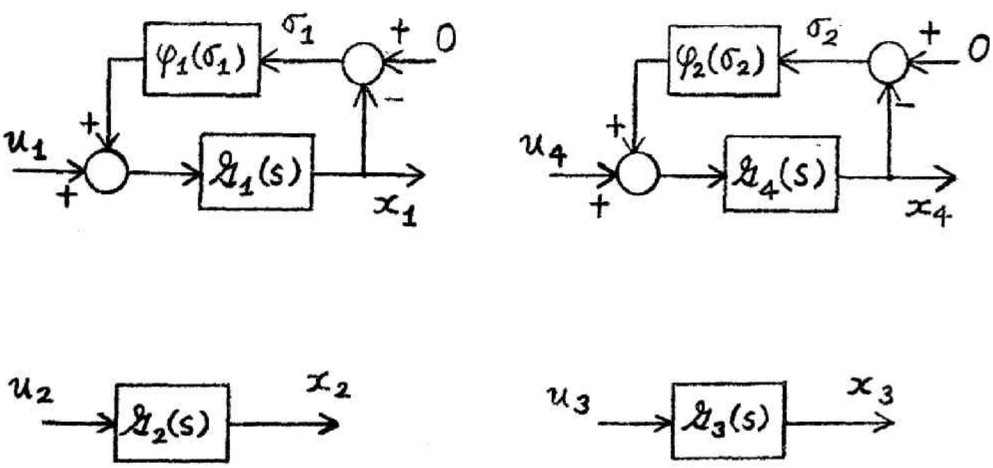


Fig. 5.10. A Composite System Equivalent to

the System of Fig. 5.9.



$$u_1^* = \varphi_1(\sigma_1^*) = \varphi_1(-x_1 - x_3)$$

$$u_2^* = \varphi_2(\sigma_2^*) = \varphi_2(-x_2 - x_4)$$

Therefore, by (5-6+) we obtain

$$|u_1^* - \varphi_1(-x_1)| \leqq k_1 |x_3|$$

$$|u_2^* - \varphi_2(-x_4)| \leqq k_2 |x_2|$$

Therefore, we can express the system as Fig. 5.10. where

$$|u_1| \leqq k_1 |x_3| \quad , \quad |u_2| \leqq k_1 |x_1| + k_1 |x_3|$$

$$|u_3| \leqq k_2 |x_2| + k_2 |x_4| \quad , \quad |u_4| \leqq k_2 |x_2|$$

Fig. 5.10. is a special case of Fig. 5.5. where the constants are given by

$$n = 4$$

$$A_{12} = A_{14} = A_{24} = A_{31} = A_{41} = A_{43} = 0$$

$$A_{13} = A_{21} = A_{23} = k_1$$

$$A_{32} = A_{34} = A_{42} = k_2$$

$$b_{ij} = 0$$

$$G_{ii}(s) = G_i(s) \quad , \quad G_{ij}(s) = 0 \quad (i \neq j)$$

Therefore, the matrix A^* is given by

$$A^* = \begin{pmatrix} \frac{1}{r_1^*} & 0 & -k_1 & 0 \\ -k_1 & \frac{1}{r_2^*} & -k_1 & 0 \\ 0 & -k_2 & \frac{1}{r_3^*} & -k_2 \\ 0 & -k_2 & 0 & \frac{1}{r_4^*} \end{pmatrix}$$

where

$$r_i^* = \begin{cases} r_i & G_i(s) = \dfrac{r_i}{1+T_i s} \\ r_i \sqrt{1+\left(\dfrac{1}{\xi_i}\right)^2} & G_i(s) = \dfrac{r_i}{1+2\xi_i T_i s + (T_i s)^2} \end{cases}$$

Therefore, by Theorem 5.3. we obtain the stability condition

$$r_2^* r_3^* k_1 k_2 (1+r_1^* k_1)(1+r_4^* k_2) < 1 \qquad (5\text{-}84)$$

__Example__ 5.4.

Let us consider the case when the i-th intrinsic subsystem and the interconnection are given by Fig. 5.11., where $\varphi_i'(\sigma_i)$ is a function satisfying

$$\varphi_i'(0)=0, \quad K_i \leqq \frac{\varphi_i'(\sigma_i)}{\sigma_i} \leqq K_i' \quad (\sigma_i \neq 0) \qquad (5\text{-}85)$$

The block diagram of the system can be transformed as shown in Fig. 5.12. Then the system is regarded as

Fig. 5.11. The Block Diagram of Example 5.4.

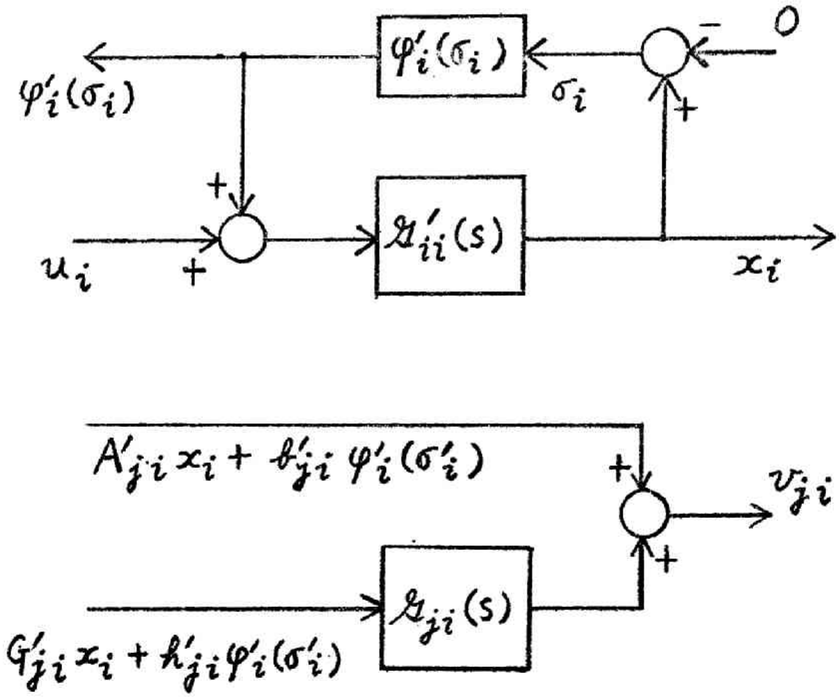


Fig. 5.12. A Composite System Equivalent to

the System of Fig. 5.11.

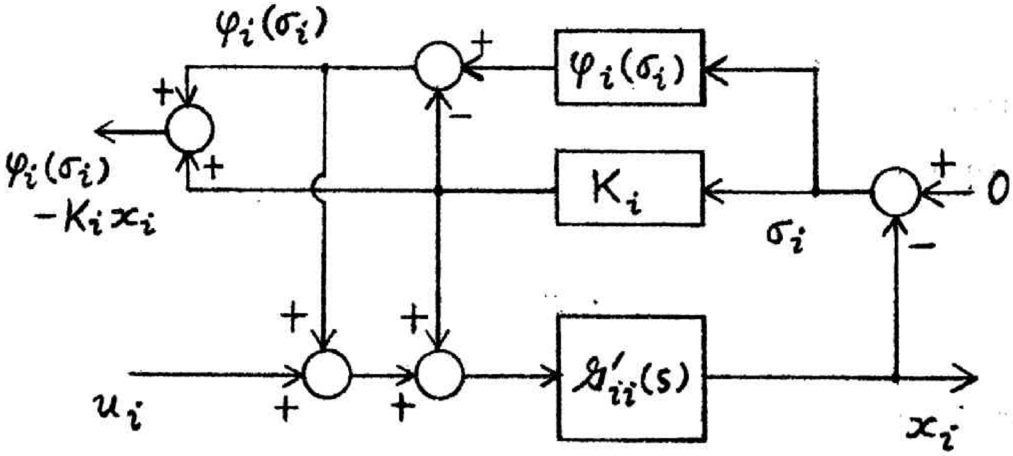
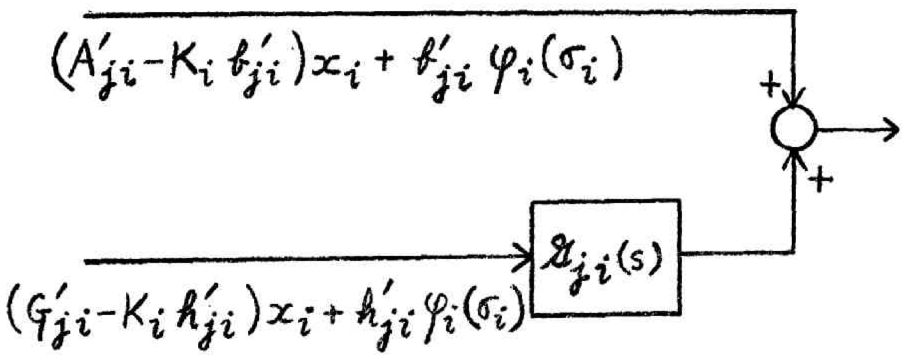


Fig. 5.12. (Continued) A Composite System

Equivalent to the System of Fig. 5.11.



having the same structure with Fig. 5.5. where

$$A_{ji} = A'_{ji} - K_i b'_{ji}$$

$$b_{ji} = b'_{ji}$$

$$G_{ji} = G'_{ji} - K_i h'_{ji}$$

$$h_{ji} = h'_{ji}$$

$$\mathcal{G}_{ii}(s) = \frac{\mathcal{G}'_{ii}(s)}{1 + K_i \mathcal{G}_{ii}(s)}$$

Here $\varphi_i(\sigma_i)$ satisfies (5-6) where

$$k_i = K'_i - K_i$$

Sec. 5.5. General Procedure of Stability

Analysis in Sampled-Data Case

In this section, a procedure of investigating the stability of the sampled-data system given by (5-7) - (5-11) will be given.

5.5.1. Stability Criterion

Theorem 5.4.

Consider the system given by (5-7) - (5-11).

Assume

(5-v) that, for each intrinsic subsystem ISS$_i$, we have an m_i-th order positive definite matrix P_{ii} and an (m_i+1)-th order positive definite matrix Q_{ii} such that

$$\left\{x_i(\tau+1)^t P_{ii} x_i(\tau+1) - x_i(\tau)^t P_{ii} x(\tau)\right\}_{(5-7),(5-8),u_i=0} \leq \begin{pmatrix} x_i \\ \varphi \end{pmatrix}^t Q_{ii} \begin{pmatrix} x_i \\ \varphi \end{pmatrix} \tag{5-86}$$

(5-vi) and that, for each connecting subsystem we have a pair of m''_{ij}-th order positive definite matrixes P_{ij} and Q_{ij} such that

$$F_{ij}^{\ t} P_{ij} F_{ij} - P_{ij} = -Q_{ij} \tag{5-87}$$

The system is a.s.i.l. if

(5-vii) the n-th order matrix $A^{\dagger}=(a^{\dagger}_{ij})$ is an M-matrix, i.e. the leading principal minor determinants of $A^{\dagger}$

are all positive (cf. Definition A.1. in Appendix).

Here a^{+}_{ij} 's are given by

$$a^{+}_{ii} = \frac{\mu_{ii}}{2\lambda_{ii}} \tag{5-88-1}$$

$$a^{+}_{ij} = -\Big\{\|C_i A_{ij}\| + k_j\|C_i b_{ij}\|\cdot\|d_j\| + \frac{\mu_{ij}}{2\lambda_{ij}}\|C_i E_{ij}\| \times(\|G_{ij}\| + k_j\|h_{ij}\|\cdot\|d_j\|) \quad i \neq j \tag{5-88-2}$$

where λ_{ii} and λ_{ij} are respectively the maximum characteristic roots of P_{ii} and P_{ij} , and μ_{ii} and μ_{ij} are respectively the minimum characteristic roots of Q_{ii} and Q_{ij} .

<u>Theorem</u> 5.5.

Consider the system given by (5-7) - (5-11). Assume (5-v) and (5-vi). The system is a.s.i.l. if (5-vii +) the n-th order matrix $A^{*}=(a^{*}_{ij})$ is an M-matrix, where a^{*}_{ij} 's are given by

$$a^{*}_{ii} = \frac{\mu_{ii}}{2\lambda_{ii}\|C_i\|} \tag{5-88-3}$$

$$a^{*}_{ij} = -\Big\{\|A_{ij}\| + k_j\|b_{ij}\|\|d_j\| + \frac{2\lambda_{ij}}{\mu_{ij}}\|E_{ij}\| \times(\|G_{ij}\| + k_j\|h_{ij}\|\cdot\|d_j\|)\Big\} \quad i \neq j \tag{5-88-4}$$

where λ_{ii} , λ_{ij} , μ_{ii} and μ_{ij} are as given in (5-vii) of Theorem 5.4.

Theorem 5.4. and 5.5. correspond to Theorem 5.1. and 5.2. in the continuous-time case, respectively. Theorem 5.5. is derived from Theorem 5.4. in the same way as the continuous-time case. Theorem 5.4. is proved in 5.5.2.

By the above theorems, we can obtain a general procedure to investigate the stability of the system given by (5-7) - (5-11), just in parallel with the continuous-time case. Concerning the establishment of the condition (5-v), the literatures [47-50] give useful results. Eq. (5-87) corresponds to eq. (5-14) in the continuous-time case. If F_{ij} has all its characteristic roots λ inside the unit circle, i.e. if

$$|\lambda| < 1$$

, eq. (5-77) has a positive definite solution P_{ij} for any positive definite righthand side Q_{ij} . (cf. [48], [9])

Here, it must be noted that, in the proof of Theorem 5.4., we use the norm N_ℓ defined as the square root of the quadratic form associated with the matrix P_{ii} or P_{ij} (cf. eq. (5-91) and (5-95)). This fact suggests us to search directly a norm N_ℓ satisfying (4-i) rather than to search the quadratic form as indicated in the above procedure. In reality, there are several researches to establish the stability of the system (the isolated subsystem in our problem) by constructing a norm N_i satisfying the condition (4-i). [9, 46, 51]. For this purpose such norms as given by

$$N_i(x_i) = \sum_{k=1}^{m_i} w_k \, |x_{ik}| \qquad w_k > 0$$

or

$$N_i(x_i) = \max_{k=1,\dots,m_i} w_k \, |x_{ik}| \qquad w_k > 0$$

are used. If such a method can be successfully applied to the isolated subsystems, it will be more profitable to use it and apply Theorem 4.1. than to follow the procedure described above. The procedure described above based on Theorem 5.4. and 5.5. is a rather circuitous way though it has advantages that the establishment of a quadratic form satisfying (5-86) is related with the character of the frequency response of the linear part [46, 52, 53] and that we have a rather general procedure of establishing the pair of P_{ii} and Q_{ii} [47, 48, 50].

5.5.2. Proof of Theorem 5.4. and a Supplement

Proof of Theorem 5.4. :

In parallel with the continuous-time case, let us regard the system given by (5-5) - (5-11) as composed of n^2 subsystems, and let us number them with the integer parameter ℓ where

$$\ell = n \times (i-1) + i \tag{5-89}$$

for the i-th intrinsic subsystem and

$$\ell = n \times (i-1) + j \qquad i \neq j \tag{5-90}$$

for the i-j connecting subsystem.

First, let us show that the assumption (4-i) is satisfied. Consider the i-th intrinsic subsystem, and put

$$N_\ell(x_i) = \{x_i^t P_{ii} x_i\}^{\frac{1}{2}} \tag{5-91}$$

$$M_\ell(x_i) = \|x_i\| \tag{5-92}$$

$$\ell = n \times (i-1) + i$$

By (5-86), we obtain

$$\begin{aligned} &\{N_\ell(x_i(\tau+1)) - N_\ell(x_i(\tau))\}_{(5\text{-}7),\,(5\text{-}8),\,u_i=0} \\ &\quad \leqq -\frac{1}{2N_\ell(x_i(\tau))} \begin{pmatrix} x_i(\tau) \\ \varphi_i(\sigma_i(\tau)) \end{pmatrix}^t Q_{ii} \begin{pmatrix} x_i(\tau) \\ \varphi_i(\sigma_i(\tau)) \end{pmatrix} \\ &\quad \leqq -\frac{\mu_{ii}}{2\sqrt{\lambda_{ii}}} \|x_i(\tau)\| \qquad \ell = n(i-1)+i \end{aligned} \tag{5-93}$$

Therefore, (4-i) is satisfied where

$$\gamma_\ell = \frac{\mu_{ii}}{2\sqrt{\lambda_{ii}}} \qquad \ell = n(i-1)+i \tag{5-94}$$

Consider the i-j connecting subsystem, and put

$$N_\ell(y_{ij}) = \{ y_{ij}{}^t \; P_{ij} \; y_{ij} \}^{\frac{1}{2}} \tag{5-95}$$

$$M_\ell(y_{ij}) = \| y_{ij} \| \tag{5-96}$$

$$\ell = n(i-1) + j$$

By (5-87), we obtain

$$\{ y_{ij}(\tau+1)^t P_{ij} \, y_{ij}(\tau+1) - y_{ij}(\tau)^t P_{ij} \, y_{ij}(\tau) \}_{(5\text{-}11),\, x_j = 0}$$
$$\leqq - y_{ij}(\tau)^t Q_{ij} \, y_{ij}(\tau) \tag{5-97}$$

Therefore, we obtain

$$\{ N_\ell(y_{ij}(\tau+1)) - N_\ell(y_{ij}(\tau)) \}_{(5\text{-}11),\, x_j = 0}$$
$$\leqq - \frac{\mu_{ij}}{2\sqrt{\lambda_{ij}}} \| y_{ij} \| \qquad \ell = n(i-1) + j \tag{5-98}$$

Therefore, (4-i) is satisfied where

$$\gamma_\ell = \frac{\mu_{ij}}{2\sqrt{\lambda_{ij}}} \qquad \ell = n(i-1) + j \tag{5-99}$$

Now, let us show that the assumption (4-ii) is satisfied. The equations (5-7) -(5-11) can be rewritten

$$x_i(\tau+1) = A_{ii} \, x_i(\tau) + b_{ii} \, \varphi_i(d_i{}^t x_i(\tau))$$
$$+ \sum_{j;j\neq i} \{ C_i A_{ij} x_j(\tau) + C_i b_{ij} \varphi_j(d_j{}^t x_j(\tau))$$
$$+ \sum_{j;j\neq i} C_i E_{ij} \, y_{ij}(\tau) \tag{5-100}$$

$$y_{ij}(\tau+1) = F_{ij}\, y_{ij}(\tau) + G_{ij}\, x_j(\tau) + h_{ij}\, \varphi_j(d_j^{\,t} x_j(\tau)) \tag{5-101}$$

Referring to eqs. (5-91), (5-92), (5-95) and (5-96) which give the definition of N_ℓ and M_ℓ, we obtain that (4-ii) is satisfied with

$$\varepsilon_{\ell\ell'} = \sqrt{\lambda_{ii}}\ \{\|C_i A_{ij}\| + \|C_i b_{ij}\| \cdot k_j \|d_j\|\}$$
$$\ell = n(i-1)+i\,,\ \ell' = n(j-1)+j\,,\ i \neq j \tag{5-102-1}$$

$$\varepsilon_{\ell\ell'} = \sqrt{\lambda_{ii}}\ \|C_i E_{ij}\|$$
$$\ell = n(i-1)+i\,,\ \ell' = n(i-1)+j\,,\ i \neq j \tag{5-102-2}$$

$$\varepsilon_{\ell\ell'} = \sqrt{\lambda_{ij}}\ \{\|G_{ij}\| + \|h_{ij}\|\ k_j \|d_j\|\}$$
$$\ell = n(i-1)+j\,,\ \ell' = n(j-1)+j\,,\ i \neq j \tag{5-102-3}$$

$\varepsilon_{\ell\ell'} = 0$ for the other pairs of ℓ and ℓ', $\ell \neq \ell'$

(5-102-4)

Therefore, by Theorem 4.1., the system is a.s.i.l. if the matrix $A = (a_{\ell\ell'})$ is an M-matrix, where

$$a_{\ell\ell} = \frac{\mu_{ij}}{2\sqrt{\lambda_{ij}}} \qquad \ell = n(i-1)+j \tag{5-103-1}$$

$$a_{\ell\ell'} = -\,\varepsilon_{\ell\ell'} \qquad \ell \neq \ell' \tag{5-103-2}$$

By Theorem A.3. in Appendix, the above condition is equivalent to the condition that

(5-viii) the matrix $A'=(a'_{ij})$ is an M-matrix where

$$a'_{\ell\ell} = \frac{\mu_{ij}}{2\lambda_{ij}} \qquad \ell = n(i-1)+j \qquad (5\text{-}104\text{-}1)$$

$$a'_{\ell\ell'} = -\frac{\varepsilon_{\ell\ell'}}{\sqrt{\lambda_{ij}}} \qquad \ell' \neq \ell,\ \ell = n(i-1)+j \qquad (5\text{-}104\text{-}2)$$

The above condition (5-viii) is equivalent to the condition (5-vii). (This can be proved just in the same way as the proof of the equivalence of (5-iv) and (5-iii).)

(Q.E.D.)

In the above proof, we can replace the relation (5-93) with

$$\left\{N_\ell(x_i(\tau+1)) - N_\ell(x_i(\tau))\right\}_{(5\text{-}7),(5\text{-}8),u_i=0} \leq -\eta_{ii}\,\|x_i(\tau)\|$$

where η_{ii} is a positive number such that

$$\begin{pmatrix} x_i \\ \varphi \end{pmatrix}^t Q_{ii} \begin{pmatrix} x_i \\ \varphi \end{pmatrix} \leq \eta_{ii}\left\{x_i^t P_{ii} x_i\right\}^{\frac{1}{2}} \|x_i\| \qquad (5\text{-}105\text{-}1)$$

We can also replace (5-98) with a similar relation. Therefore, we obtain the next supplement.

<u>Supplement to Theorem 5.4. and 5.5.</u>

The number μ_{ij}/λ_{ij} in (5-88-1) - (5-88-4) can be replaced by

$$\frac{\eta_{ij}}{\sqrt{\lambda_{ij}}}$$

where η_{ii} and η_{ij} are positive numbers which satisfy (5-105-1) and

$$y_{ij}{}^t Q_{ij} y_{ij} \leqq \eta_{ij} \{ y_{ij}{}^t P_{ij} y_{ij} \}^{\frac{1}{2}} \| y_{ij} \| \quad i \neq j \tag{5-105-2}$$

Especially, when $Q_{ij} = P_{ij}$, we can put

$$\eta_{ij} = \sqrt{\mu_{ij}} \tag{5-106}$$

Sec. 5.6. An Example of Sampled-Data Systems

In this section, an example of the sampled-data composite system is studied.

Example 5.5.

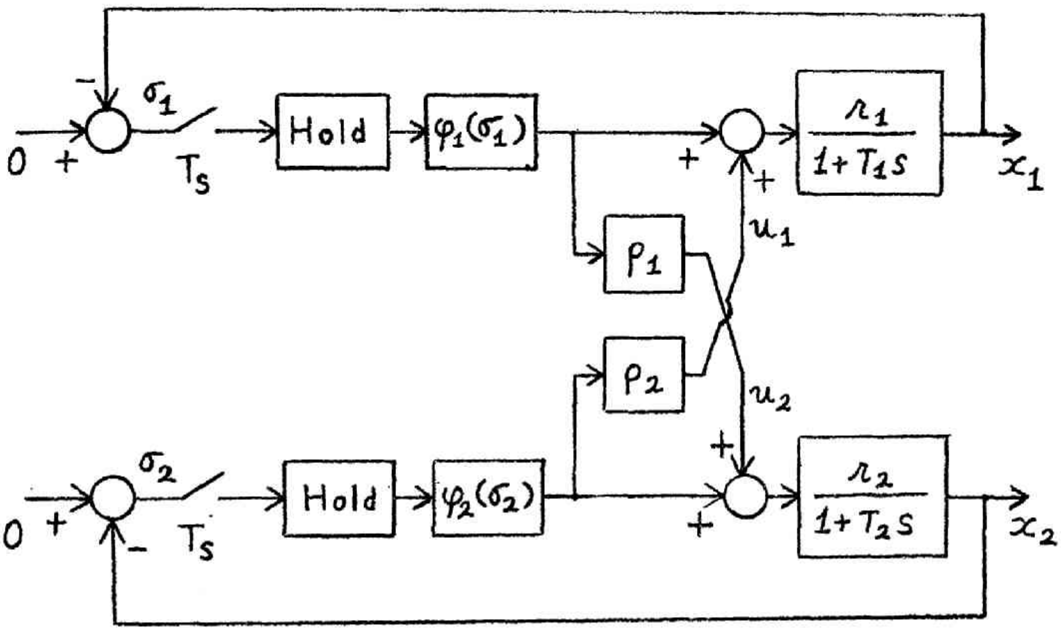
Consider the system given in Fig. 5.13., where the functions $\varphi_i(\sigma_i)$ $(i=1,2)$ satisfy (5-6), and T_S is the sampling period. The system is described by

$$\begin{aligned} x_i(\tau+1) = A_i x_i(\tau) &+ \kappa_i (1-A_i) \varphi_i(\sigma_i(\tau)) \\ &+ \kappa_i (1-A_i) u_i(\tau) \end{aligned} \tag{5-107-1}$$

$$i = 1, 2$$

Fig. 5.13. The Block Diagram of Example 5.5.

$r_1 > 0, \ T_1 > 0, \ r_2 > 0, \ T_2 > 0$



$$\sigma_i(\tau) = -x_i(\tau) \qquad (5\text{-}108\text{-}i)$$

$$i = 1, 2$$

$$u_1(\tau) = \rho_2 \, \varphi_2(\sigma_2(\tau)) \qquad (5\text{-}109\text{-}1)$$

$$u_2(\tau) = \rho_1 \, \varphi_1(\sigma_1(\tau)) \qquad (5\text{-}109\text{-}2)$$

$$A_i = \exp\left(-\frac{T_s}{T_i}\right) \qquad i = 1, 2 \qquad (5\text{-}110)$$

where $x_1(\tau)$, $x_2(\tau)$, $\sigma_1(\tau)$ and $\sigma_2(\tau)$ are the values of x_1, x_2, σ_1 and σ_2 at the time

$$t = \tau T_s$$

The above equations are a special case of the equations (5-7) - (5-11). The system is composed of two subsystems, the first and the second of which are described respectively by (5-107-1) and (5-108-1) and by (5-107-2) and (5-108-2). The isolated i-th subsystem ($i=1,2$) is a.s.i.l. for any $\varphi_i(\sigma_i)$ satisfying (5-6) if and only if

$$r_i k_i (1-A_i) - A_i < 1 \qquad (5\text{-}111)$$

In the following, let us assume (5-111) for both subsystems.

First, let us use Theorem 4.1. Put

$$N_i(x_i) = M_i(x_i) = |x_i| \qquad i = 1, 2$$

Then, the assumption (4-i) is satisfied where

$$\gamma_i = \begin{cases} 1-A_i & k_i r_i < \dfrac{2A_i}{1-A_i} \\ \\ 1+A_i - k_i r_i (1-A_i) & \dfrac{2A_i}{1-A_i} < k_i r_i < \dfrac{1+A_i}{1-A_i} \end{cases} \qquad (5\text{-}112)$$

The assumption (4-ii) is satisfied where

$$\varepsilon_{12} = r_1 (1-A_1) k_2 |\rho_2|$$

$$\varepsilon_{21} = r_2 (1-A_2) k_1 |\rho_1|$$

Therefore, we obtain the stability condition

$$|\rho_1 \rho_2| \leqq \frac{\gamma_1 \gamma_2}{k_1 k_2 r_1 r_2 (1-A_1)(1-A_2)} \qquad (5\text{-}113)$$

where γ_1 and γ_2 are given by (5-112).

Now, let us apply Theorem 5.4. to the above system. We can easily obtain

$$\left\{x_i(\tau+1)^2 - x_i(\tau)^2\right\}_{(5\text{-}107\text{-}i),\,(5\text{-}108\text{-}i),\,u_i=0}$$

$$\leqq -2\gamma_i' \, x_i(\tau)^2$$

$$\gamma_i' = \gamma_i \left(1 - \frac{\gamma_i}{2}\right) < \gamma_i$$

where γ_i are given by (5-112). Therefore, by Theorem 5.4. we obtain the stability condition

$$|\rho_1 \rho_2| \leqq \frac{\gamma_1' \gamma_2'}{k_1 k_2 r_1 r_2 (1-A_1)(1-A_2)} \qquad (5\text{-}114)$$

The condition (5-114) is more conservative than the condition (5-113). This is because, in obtaining (5-114), we passed the unnecessary step of constructing a Lyapunov function of the quadratic form. This example shows that Theorem 5.4. is not recommendable if we can construct a Lyapunov function of the "norm" type directly.

This example will be studied again in 5.7.2. (at. p. 141), where we obtain a less conservative condition.

Sec. 5.7. Improvement of the Stability Criterion

In this section, it is shown that the stability condition obtained in the preceding sections can be improved by use of general positive functions $M_i(x_i)$ in place of $\|x_i\|$ as indicated in Theorem 3.2. or 4.1.

5.7.1. Continuous-Time Case (Supplement to Theorem 5.3.)

In order to obtain Supplement to Theorem 5.1. and 5.2., we used already Theorem 3.2. But, there, we only used the replacement of $\nabla_i v_i$ with modified($\nabla_i v_i$), and we did not apply the usage of general positive definite functions $M_i(x_i)$. If we examine a little carafully the assumption (5-i), we can see that it is possible to induce the relations (3-2-g), (3-3-g) and (3-4-g) by using the functions $M_i(x_i)$ of the form

$$M_i(x_i) = \left\{ \begin{pmatrix} x_i \\ \varphi_i \end{pmatrix}^t R_i \begin{pmatrix} x_i \\ \varphi_i \end{pmatrix} \right\}^{\frac{1}{2}} \qquad (5\text{-}115)$$

where R_i is a positive semi-definite matrix. Therefore, if we can assure that $M_i(x_i)$'s given by (5-115) are positive definite, we can apply Theorem 3.2. We have much possibility of obtaining better (less conservative) stability criterion by the use of $M_i(x_i)$ as given by (5-115). However, it is tremendously difficult to find out generally the best $M_i(x_i)$ among all possible $M_i(x_i)$'s so that we can obtain the least conservative stability criterion. Therefore, here we make only one trial to improve the stability criterion by using a special $M_i(x_i)$, and we confine our study to the case in which the system has the form as given in Fig. 5.5. and the transfer function of the intrinsic subsystem is the first order one given by (5-41).

Supplement to Theorem 5.3.

In the system of Fig. 5.5., assume that, for a certain value of i, we have

$$\mathcal{G}_{ii}(s) = \frac{\lambda_{ii}}{1+T_{ii}s} \qquad \lambda_{ii}>0\ ,\ T_{ii}>0$$

$$\varphi_i(\sigma_i) \neq 0 \quad \text{for} \quad \sigma_i \neq 0 \tag{5-116}$$

$$\int_0^{\sigma_i} \varphi_i(\xi)\,d\xi \to \infty \quad \text{for} \quad |\sigma_i| \to \infty \tag{5-117}$$

$$A_{ji} = G_{ji} = 0 \qquad j=1,\cdots,n\ ;\ j\neq i \tag{5-118}$$

Then, the element a^*_{ii} in Table 5.1. can be replaced by

$$a^*_{ii} = \frac{1}{\lambda_{ii}} + k_i \tag{5-119}$$

Proof : First note that the value of i is fixed throughout this proof.

Put

$$v_i(x_i) = \int_0^{-x_i} \varphi(\xi)\, d\xi \tag{5-120}$$

$$M_i(x_i) = \frac{1}{k_i} |\varphi(-x_i)| \tag{5-121}$$

By (5-117), the function $v_i(x_i)$ satisfies the relation (3-1), and by (5-116) the function $M_i(x_i)$ is positive definite. In order to prove the supplement, we must refer to 5.2.2. (Proof of Theorem 5.1.), 5.3.2. (First Order Intrinsic Subsystem) and 5.3.4. (Connection between Intrinsic Subsystems). To prove the supplement, first we show that, for the i-th intrinsic subsystem,

$$\left(\frac{dv_i}{dt}\right)_{(5\text{-}43),\, u_i=0} \leqq -\frac{k_i r_{ii}}{T_{ii}}\left(k_i + \frac{1}{r_{ii}}\right) M_i(x_i)^2 \tag{5-122}$$

$$\left|\frac{\partial v_i}{\partial x_i}\right| \leqq k_i M_i(x_i) \tag{5-123}$$

hold in place of (5-44) and (5-45). Then, we show that the interaction from ℓ_1-th $(\ell_1 = n(i-1)+1)$ subsystem to the ℓ-th $(\ell = 1, \cdots, n^2;\ \ell \neq \ell_1)$ subsystem is evaluated by the relation

$$\| \mathbb{f}_{\ell\ell_1}(x_i, t) \| \leqq \varepsilon_{\ell\ell_1} M_i(x_i) \tag{3-4-g}$$

where $\varepsilon_{\ell\ell_1}$'s are same as given by (5-35). By these relations, we can easily conclude the supplement just in the same way as the proof of Theorem 5.1. and 5.3.

By differentiating (5-120) along the solution of (5-43), and considering the relation (5-6), we easily obtain (5-122). The relation (5-123) is evident from

$$\frac{\partial v_i}{\partial t} = -\varphi_i(-x_i) \tag{5-124}$$

Next, let us consider the interaction from the i-th intrinsic subsystem to the other subsystems. The interaction to the j-th intrinsic subsystem $(j \neq i)$ is given by the term $b_{ji}\varphi_i(-x_i)$. Therefore, the relation (3-4-g) holds where

$$\varepsilon_{\ell\ell_1} = k_i |b_{ji}| \tag{5-125}$$

$$\ell = n(j-1)+j \ , \ j \neq i \ , \ \ell_1 = n(i-1)+i$$

The interaction to the i-j connecting subsystem $(j \neq i)$ is given by the term $h_{ji}\varphi_i(-x_i)$. Therefore, the relation (3-4-g) holds where

$$\varepsilon_{\ell\ell_1} = k_i |h_{ji}| \tag{5-126}$$

$$\ell = n(j-1)+i \ , \ j \neq i \ , \ \ell_1 = n(i-1)+i$$

If we note (5-118), we can see that the value of $\varepsilon_{\ell\ell_1}$ given by (5-125) and (5-126) are same with the values given by (5-35). (Q.E.D.)

Here, in order to show the effectiveness of the above supplement, let us consider Example 5.2. again.

Example 5.2. (continued)

Let us use the Supplement to Theorem 5.3. Then, instead of (5-81), we obtain the stability condition

$$|\rho_1 \rho_2| < \left(\frac{1}{r_1} + k_1\right)\left(\frac{1}{r_2} + k_2\right)\frac{1}{k_1 k_2}$$
$$= \left(1 + \frac{1}{r_1 k_1}\right)\left(1 + \frac{1}{r_2 k_2}\right) \qquad (5\text{-}127)$$

If we compare (5-127) with (5-83), we see that the condition obtained here differs from the condition for the linearized system only by the region

$$\rho_1 \rho_2 \leqq -\left(1 + \frac{1}{r_1 k_1}\right)\left(1 + \frac{1}{r_2 k_2}\right) \qquad (5\text{-}128)$$

This is a great improvement compared with the condition (5-81). The reason of the success of the application of Supplement to Theorem 5.3. to this example is that, by use of the supplement, we can utilize the peculiarity of the structure of the system that the interactions depend only upon the factor $\varphi_i(\sigma_i)$.

In this section, we only made a restricted trial to improve the stability criterion by use of the functions $M_i(x_i)$ given by (5-115). The results obtained here show that this kind of trial is very effective when the interaction depends only upon the function $\varphi_i(\sigma_i)$. The discussion made here can be extended to the cases of second or higher order transfer functions though the results become rather complex.

5.7.2. Sampled-Data Case

The same kind of improvement as stated in 5.7.1. can be made also in the sampled-data case. Here, let us study Example 5.5. again and see that the stability condition is improved by use of general positive definite functions $M_i(x_i)$.

Example 5.5. (continued)

First, let us assume the same condition (5-116) employed in Supplement to Theorem 5.3.

$$\varphi_i(\sigma_i) \neq 0 \quad \text{for} \quad \sigma_i \neq 0 \tag{5-116}$$

Let us apply Theorem 4.1. by putting

$$N_i(x_i) = |x_i| \;,\; M_i(x_i) = |\varphi_i(\sigma_i)| \tag{5-129}$$

Then, the assumption (4-i) is satisfied where

$$\gamma_i = \begin{cases} (1-A_i)\left(r_i + \dfrac{1}{k_i}\right) & r_i k_i \leqq \dfrac{A_i}{1-A_i} \\[2ex] \dfrac{1+A_i}{k_i} - r_i(1-A_i) & \dfrac{A_i}{1-A_i} < r_i k_i < \dfrac{1+A_i}{1-A_i} \end{cases} \tag{5-130}$$

The assumption (4-ii) is satisfied where

$$\varepsilon_{12} = r_1 (1-A_1) |\rho_2|$$

$$\varepsilon_{21} = r_2 (1-A_2) |\rho_1|$$

Therefore, we obtain the stability condition

$$|\rho_1 \rho_2| < \frac{\gamma_1 \gamma_2}{r_1 r_2 (1-A_1)(1-A_2)} \tag{5-131}$$

where γ_1 and γ_2 are given by (5-130).

Here, let us compare (5-113) and (5-131) for the case

$$r_i k_i \leqq \frac{A_i}{1-A_i} \tag{5-132}$$

In this case, the condition (5-113) becomes

$$|\rho_1 \rho_2| < \frac{1}{r_1 r_2 k_1 k_2} \tag{5-113'}$$

and (5-131) becomes

$$|\rho_1 \rho_2| < \left(1+\frac{1}{r_1 k_1}\right)\left(1+\frac{1}{r_2 k_2}\right) \tag{5-131'}$$

Clearly, the condition (5-131) is less conservative than (5-113).

From the result of the above example, we can guess that the functions $M_i(x_i)$ of the type

$$M_i(x_i) = \sum_{k=1}^{m_i} M'_{ik} |x_{ik}| + M''_i |\varphi_i(\sigma_i)| \tag{5-133}$$

$$M'_{ik} \geqq 0, \quad M''_i \geqq 0$$

or of the type

$$M_i(x_i) = \max \left(M'_{ik} |x_{ik}|, M''_i |\varphi_i(\sigma_i)| \right) \tag{5-134}$$

$$k = 1, \ldots, m_i \; ; \quad M'_{ik} \geqq 0, \quad M''_i \geqq 0$$

may contribute to the improvement of the stability condition.

Chapter 6 Conclusion

Some remarks on the main theorems of Chapter 3 and Chapter 4 will be made in the first section. Then, remarks on the practical application are made. Lastly, main results and future problems are summarized.

Sec. 6.1. Remarks on the Main Theorems

First, Theorem 3.1. (the main theorem in the continuous time case) and Theorem 4.1. (the main theorem in the sampled-data case) are compared. Then, possibility of obtaining a less conservative condition is discussed.

6.1.1. Comparison of the Main Theorems

Theorem 3.1. (at p. 17) gives a stability condition for continuous-time composite systems and Theorem 4.1. (at p. 54) gives a stability condition for sampled-data systems. In both cases, the stability condition is expressed as a requirement that a certain matrix A be an M-matrix. The matrix A is given in a similar formula (eqs. (3-5) and (4-5)) in both cases. Especially, if a quadratic form Lyapunov function is given for each subsystem, the stability conditions of both cases have a same form as given by eqs. (5-15) (at. p. 88) and (5-88) (at p. 125). Thus, the results of the two theorems have a similar property in common.

However, the two theorems differ in their assumptions (3-i) and (4-i). In (3-i), the Lyapunov function $v_i(x_i, t)$ can have an arbitrary form under the constraints (3-1), (3-2) and (3-3) (or (3-1), (3-2-g) and (3-3-g) by Theorem 3.2.). On the other hand, in (4-i), the Lyapunov function $N_i(x_i)$ must be a norm (or, by Theorem 4.2., a positive definite function satisfying the triangular inequality (4-20)). The above difference has its origin in the schemes employed for the proofs of the two theorems. In the proof of Theorem 3.1. (i.e. the proof of Lemma 3.1. at p. 26), we use the positive definiteness of a quadratic form to establish the negative definiteness of the derivative of the function $V(x)$ given by (3-23) along the solution of the composite system. In order to obtain this quadratic form, the relations (3-2) and (3-3) are necessary (cf. p. 26). On the other hand, in the proof of Theorem 4.1. (i.e. the proof of Lemma 4.1. at p. 58), we separate the interaction terms by the other subsystems from the self-restoration term of the i-th subsystem in the difference of $N(x)$ given by (4-14) along the solution of the composite system, and by this separation we establish the negative definiteness of the difference. For this purpose, the triangular inequality is necessary.

The above two schemes of the proofs are respectively related with the essential characters of the continuous-time and the sampled-data systems, and it is difficult to exchange the schemes. In reality, the norm $N(x)$ given by (4-14) is not differentiable at $x_i = 0$ (for some i) with respect to the components of x . This causes a difficulty in using $N(x)$ as a Lyapunov function of the continuous-time system given by differential equations (cf. [35]). On the other hand, an representative example of the function $v_i(x, t)$ satisfying (3-i) is a quadratic form of x_i. If we use a quadratic form $v_i(x_i, t)$ of x_i as a Lyapunov function of the isolated subsystem in the sampled-data case, we will have many cross-terms of the interactions by the other subsystems in the difference of the function $v_i(x_i, t)$ along the solution of the composite system. This causes a difficulty in establishing the negative definiteness of the difference of $V(x, t)$ given by (3-23), which is necessary to establish a.s.i.l. of the composite system [33].

6.1.2. On the Possibility of Less Conservative Stability Conditions

The discussion of this item is mainly motivated by the theoretical interest. In the following, we show that the condition (3-iii) of Theorem 3.1. and the condition (4-iii) of Theorem 4.1. are respectively the sharpest stability condition which can be given with the information contained in (3-i) and (3-ii), or in (4-i) and (4-ii).

First, let us consider Theorem 3.1. In Theorem 3.1. the stability condition (3-iii) is given only using the information given by (3-i) and (3-ii). In other words, only by knowing the existence of the functions $v_i(x_i, t)$ $\alpha_i(\|x_i\|)$ and $\beta_i(\|x_i\|)$ and the values of the constants γ_i, δ_i and ε_{ij}, we can establish a.s.i.l. of the system by the condition (3-iii). Here, we naturally have a question : "Can we obtain any other stability condition, which uses only the information given by (3-i) and (3-ii) and which is less conservative than the condition (3-iii) ?" The answer to this question is negative, as shown in the following.

<u>Proposition</u> 6.1.

Let Ω a set of conditions on positive numbers

γ_i, $\delta_i (i = 1, ---, n)$ and $\varepsilon_{ij} (i, j = 1, ---, n; i \neq j)$.
Assume that the satisfaction of Ω implies a.s.i.l. of any system described by a set of differential equations of the form (2-1) which satisfies the assumptions (3-i) and (3-ii). Then, Ω implies (3-iii).

Proof : Consider the simplest linear system with positive connections given by (3-6) and put

$$l_{ii} = -\frac{\gamma_i^*}{\delta_i^*}, \; l_{ij} = \varepsilon_{ij}^* \; (i \neq j) ; \; \gamma_i^*, \delta_i^*, \varepsilon_{ij} > 0 \tag{6-1}$$

By putting

$$v_i(x_i, t) = \frac{1}{2} \delta_i^* x_i^2 \tag{6-2}$$

the simplest linear system with the coefficients given by (6-1) satisfies the assumptions (3-i) and (3-ii), where

$$\gamma_i = \gamma_i^* \;,\; \delta_i = \delta_i^* \;,\; \varepsilon_{ij} = \varepsilon_{ij}^* \tag{6-3}$$

Here, assume that Ω is satisfied with the values of γ_i, δ_i and ε_{ij} given by (6-3). Then, by the assumption on Ω, the simplest linear composite system with the coefficients given by (6-1) is a.s.i.l. Therefore, by the fact that (3-iii) is necessary for a.s.i.l. of the simplest linear composite system with positive connections (at p. 19), (3-iii) is satisfied with the values of γ_i, δ_i and ε_{ij} given by (6-3). (Q.E.D.)

The above proposition tells that the condition (3-iii) is the sharpest stability condition which can be given with the information contained in (3-i) and (3-ii).

In the same way, we can prove the next proposition on Theorem 4.1.

Proposition 6.2.

Let Ω a set of conditions on positive numbers γ_i, δ_i $(i = 1, ---, n)$ and ε_{ij} $(i, j = 1, ---, n\ ;\ i \neq j)$ Assume that the satisfaction of Ω implies a.s.i.l. of any system described by a set of difference equations of the form (2-9) which satisfies the assumptions (4-i) and (4-ii). Then, Ω implies (4-iii).

Sec. 6.2. Remarks on the Practical Application

The results obtained in the preceding chapters can be applied for the investigation of the staability of multidimensional nonlinear systems only if we can construct a composite system model as given in Chapter 2 and if the individual subsystems contain only a few nonlinearities. Here, it is not always necessary to construct a composite system model in a narrow sense as given by eq. (2-1) (at p. 10) or eq. (2-9) (at. p. 14), but it is enough to construct a composite system model in a wide sense as given by eqs. (3-40) and (3-41) or by eqs. (4-29) and (4-30). (cf. 3.4.1. in Sec. 3.4. at pp. 34-38 and also cf. 4.3.2. in Sec. 4.3. at p. 67) For a system of engineering interest, it is almost always possible to construct a composite system model in a wide sense if the nonlinearities appearing in the connecting part satisfy the property (3-38) (cf. 3.4.1. in Sec. 3.4. and Example 5.3. at p. 118. Note that eq. (3-38) is same with eq. (5-6+).) In this sense, we can say the results obtained in the preceding chapters can be applied to complex multidimensional nonlinear systems of engineering interest quite generally, and that there is not a great restriction on its application.

In the following, as a typical example of the practical application of our theorems, we study the multilevel and decentralized control methods of a large scale system in a little detailed manner.

In controling a large scale system, it is desirable to construct a centralized control system* from the standpoint of optimizing the overall performance of the

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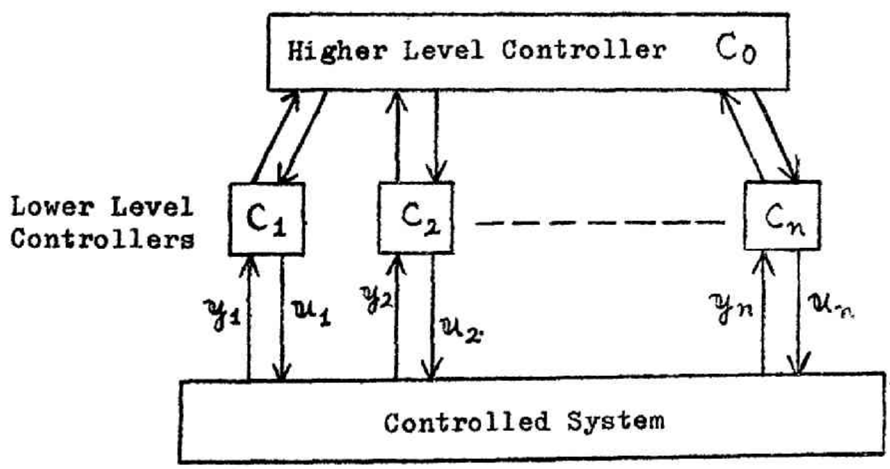
* Here, we refer to a case with the word "centralized control" if all the actuators involved in the system are operated by one central computer.

system. However, the centralized control system is not always profitable if we consider the cost necessary for the construction of a complete communication system connecting all actuators and observers to the central computer, the requirements on the ability of the central computer and the trouble which might be caused by a failure in the communication system or in the central computer. Hereby, we arrive at the "multilevel control" method where we adopt some decentralization of the control. [62]

For instance, the two-level control system can be figurized as Fig. 6.1. Here, each lower level controller C_i controls the input u_i only by observing y_i. The higher level controller C_0 determines

Fig. 6.1. A Two-Level Control System



periodically the reference values and possibly some parameters of the lower level controllers by using all the information obtained about the system. In such a two-level control system, the lower level controllers work for the appropriate operation of the system in a short period and for the local optimization of the system, while the higher level controller is responsible for the overall optimization of the system. Considering such rolls of each controller, it is desirable, and often necessary, to design the system so that the system is stable even if the higher level controller is removed and the control of the system is left to the lower level controllers. This assertion is also verified by the consideration of the facts that the higher level controller must make a complex calculation and that, therefore, the higher level controller requires comparatively long time for computation and has more possibility of faults.

For the study of such stability problems, which appear in multilevel control systems as mentioned above, the results obtained in the preceding chapters will be greatly helpful. In reality, if we regard each lower level controller together with some corresponding part of the system as one subsystem, we obtain a composite system model for a multilevel control system. Then, our theorems can give a stability condition and an estimate of the transient behavior of the whole system. Especially, the latter result will be useful not only for the analysis but also for the synthesis of the system.

The multilevel control method can be understood as a hybrid of the centralized control strategy and the decentralized control strategy*. Thereby, the multilevel control method includes a variety of cases according to the frequency of the instructions given by the higher level controller. If the higher level controller gives instructions very often, the system becomes near to the centralized control system, and if the higher level controller gives instructions very rarely, the system becomes near to the decentralized control system.

As an example of practical problems, for which the multilevel control method is proposed in variety

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* Here, we refer to a case by the word "decentralized control" if the actuators and observers in the system are devided into several groups and actuators belonging to one group are controled by an individual computer which operates only knowing the values obtained from the observers of that group.

of forms, we can refer to the voltage and reactive power control problem in the electric power systems. For this problem, several authors have proposed variety of strategy from a purely centralized control method to a decentralized control method [54-61]. In this case, if we adopt a decentralized control strategy, the stability of the whole system becomes an important problem [58-60]. The method of composite system stability analysis proposed in this thesis was really applied to the stability problem appearing in the decentralized control of the power system voltage and reactive power, and helpful results were obtained [33].

Sec. 6.3. Main Results and Future Problems

Let us summarize the main results obtained in the preceding chapters and list up the problems remaining unsolved.

In Chapter 3, we obtained Theorem 3.1. (at p. 17) which gives a stability criterion for continuous-time composite systems. The theorem gives the stability criterion in a very simple form that the principal minor determinants of a certain matrix A be all positive. The theorem can be interpreted as a mathematical expression of the assertion : "if the individual subsystems are stable and if the interconnection among the subsystems is sufficiently small compared with the stability of the subsystems, the whole system is stable."

In Sec. 3.1., it was shown that Theorem 3.1. gives a necessary and sufficient condition for a certain type of linear system, i.e. the simplest linear composite system with positive connections (defined at p. 19). In Sec. 3.2., the assumptions of Theorem 3.1. were weakened (Theorem 3.2. at p. 28). In Sec. 3.3. the theorem is compared with the result previously obtained by Bailey [17] and it was shown that Theorem 3.1. generally gives a less conservative condition than Bailey's theorem. This fact was exemplified in Sec. 3.6. by Example 3.2. (at p. 49). In Sec. 3.4., Theorem 3.1. was generalized to the case in which the interactions to individual subsystems are nonlinearly superposed (Theorem 3.4. at p. 36) and also to the case of the stability of an invariant set (Theorem 3.5. at p. 38).

In Chapter 4, we obtained Theorem 4.1. (at p. 54) which gives a stability criterion for sampled-data composite systems. This theorem has a similar form and interpretation as Theorem 3.1. in the continuous-time case. In Sec. 4.1., it was shown that the theorem gives a necessary and sufficient condition for a certain type of linear sampled-data system, i.e. the simplest sampled-data composite system with positive connections (defined at p. 57). The assumptions of Theorem 4.1. are weakened in Sec. 4.2. (Theorem 4.2. at p. 61) In Sec. 4.3., Theorem 4.1. is generalized to the case of systems with irregularly operating controllers (Theorem 4.3. at p. 64), to the case in which the interactions to a subsystem are nonlinearly superposed (Theorem 4.5. at p. 67), and also to the case of the stability of an invariant set (Theorem 4.6. at p. 67).

The two theorems mentioned above, i.e. Theorem 3.1. and 4.1., together with their generalization enable us to investigate the stability of complex multidimensional systems with many nonlinearities in a systematic way. In order to investigate an engineering system by applying the above two theorems, first we must construct a composite system model of the engineering system (cf. Sec. 6.2.). If we obtain the composite system model in the narrow sense as given by eqs. (2-1) or (2-9) (Chapter 2, at p. 10 and 14) or the composite system model in a wide sense as given by eqs. (3-40) and (3-41) (at p. 36) or eqs. (4-29) and (4-30) (at p. 67), we can know the stability property of the whole system by investigating the individual subsystems and the connections among them.

In Chapter 5, the above procedure of stability investigation was carried into execution on a system composed of subsystems containing a single nonlinearity (Theorem 5.1. and 5.2. at p. 87 and Theorem 5.3. and 5.4. at p. 124). Especially, a continuous-time system composed of first and second order subsystems was investigated in detail in Sec. 5.3. and 5.7. Theorem 5.3. (at p. 113), together with its supplement (at p. 136), gives a straightforward stability criterion for such a system as composed of first and second order subsystems containing single sector nonlinearity (Fig. 5.5. at p. 111).

The studies on the simplest linear composite system in Sec. 3.1. (at p. 19), on Example 3.2. in Sec. 3.6. (at p. 49) and on Example 5.1. and 5.2. in Sec. 5.4. and 5.7. (at p. 114, 117 and 140) show us that Theorem 3.1. gives a pretty sharp stability condition, if carefully used, except the point that it neglects the phase-shifting effect (or, in a simpler word, the sign) of the interconnection. The consideration in 6.1.2. of Sec. 6.1. shows that Theorem 3.1. is the sharpest condition which can be given with the information contained in the assumptions (3-i) and (3-ii). Theorem 4.1. has a parallel property.

In addition to the investigation of the mere stability of the system, Theorem 3.6. given in Sec. 3.5. (at p. 39) and Theorem 4.7. given in Sec. 4.4. (at p. 69) enable us to obtain an explicit estimate of the damping coefficient (or the decaying factor) of the system.

Thus, by the researches reported here, we have attained a systematic way of investigating the dynamical property of complex high-dimensional systems containing many nonlinearities, which were difficult to be analyzed in general before. However, the following problems are still remaining to be solved in the future.

First, Table 5.1. given in Sec. 5.3. (at p. 112) only provides constants for first and second order transfer functions. To enlarge this table for more complex transfer functions is a work to be done in the future. Second, concerning the estimate of the transient behavior of the system, we did not calculated out any concrete results except Example 3.1. (at p. 47) and Example 3.2. (at p. 49). So, to provides constants for the calculation of the estimate of the transient behavior of such a general system as given in Fig. 5.5. is left. Third, it may be interesting to add some new assumptions and to try to induce a sharper stability condition for composite systems. Finally, though it is not so important from practical viewpoint, it is theoretically interesting to look for a smaller or the smallest necessary and sufficient set of conditions that the matrix A be an M-matrix. (cf. Appendix, Note 1)

Appendix M-Matrixes

Definition A.1.

An n-th order square matrix $A=(a_{ij})$ is said an M-matrix if and only if its off-diagonal elements are all non-positive

$$a_{ij} \leq 0 \qquad i \neq j \tag{A-1}$$

(A-0) and its leading principal minor determinants are all positive

$$D_k \equiv \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \qquad k = 1, \cdots, n \tag{A-2}$$

Concerning M-matrixes, we have next theorems.

Theorem A.1.

Let $A=(a_{ij})$ an n-th order square matrix with non-positive off-diagonal elements. Consider the linear equations associated with the matrix A;

$$A\boldsymbol{x} = \boldsymbol{c} \tag{A-3}$$

$$A^t \boldsymbol{y} = \boldsymbol{d} \tag{A-4}$$

Any of the next conditions (A-i) - (A-vii) is a necessary and sufficient condition that A be an M-matrix.

(A-i) The principal minor determinants of A are all positive.

(A-ii) There exists a vector $\mathbb{c} = (c_1, ---, c_n)^t$ with positive components $c_1 > 0, ---, c_n > 0$ such that eq. (A-3) has a solution $\mathfrak{x} = (x_1, ---, x_n)^t$ where $x_1 > 0, ---, x_n > 0$

(A-iii) There exists a vector $\mathbb{c} = (c_1, ---, c_n)^t$ with positive components $c_1 > 0, ---, c_n > 0$ such that eq. (A-3) has a solution $\mathfrak{x} = (x_1, ---, x_n)^t$ where $x_1 \geqq 0, ---, x_n \geqq 0$.

(A-iv) There exists a vector $\mathbb{d} = (d_1, ---, d_n)^t$ with positive components $d_1 > 0, ---, d_n > 0$ such that eq. (A-4) has a solution $\mathfrak{y} = (y_1, ---, y_n)^t$ where $y_1 > 0, ---, y_n > 0$.

(A-v) There exists a vector $\mathbb{d} = (d_1, ---, d_n)^t$ with positive components $d_1 > 0, ---, d_n > 0$ such that eq. (A-4) has a solution $\mathfrak{y} = (y_1, ---, y_n)^t$ where $y_1 \geqq 0, ---, y_n \geqq 0$.

(A-vi) The matrix A is non-singular and the elements of A^{-1} are all non-negative.

(A-vii) The real parts of the characteristic roots of A are all positive.

<u>Theorem</u> A.2.

If we increase an element of an M-matrix so as to keep the sign of the element unchanged, the value of any principal minor determinant does not decrease.

Let $A = (a_{ij})$ and $A' = (a'_{ij})$ a pair of n-th order matrixes with non-positive off-diagonal elements which satisfy the relation

$$a_{ij} \geqq a'_{ij}$$

Then, if A' is an M-matrix, A is an M-matrix.

Theorem A.3.

If we multiply a row (or a column) of an M-matrix by a positive number, we obtain another M-matrix.

The proof of Theorem A.1. is in Nikaido [63,64] and in Ostrowski [42, 43]. Theorem A.2. is proved in Ostrowski [42]. Theorem A.3. is rather evident from the condition (A-ii) or (A-iv) of Theorem A.1.

The above theorems have close relations with Frobenius-Perron's theorem (p. 278 of [22], p. 100 of [63], p. 120 of [64]) on non-negative matrixes.

Theorem A.4. (Frobenius and Perron)

Let $B=(b_{ij})$ a non-negative square matrix (a square matrix whose elements are all non-negative). Then, B has a non-negative characteristic root λ_B which has the following properties.

(A-viii) The absolute value of any characteristic root ω_B of B is not greater than λ_B.

(A-ix) A non-negative vector $x=(x_1, ---, x_n)^t$ (a vector whose components are all non-negative) is associated with the characteristic root λ_B , i.e.

$$B x = \lambda_B x \qquad x \neq 0 ; \; x_1 \geqq 0, \cdots, x_n \geqq 0 \quad \text{(A-5)}$$

(A-x) Put

$$A = \rho I - B$$

The matrix A is an M-matrix if and only if

$$\rho > \lambda_B$$

(A-xi) If there is a non-negative vector y such that

$$B y \geqq \mu y \quad , \; y \neq 0$$

then

$$\lambda_B \geqq \mu$$

Here, the inequality between vectors means that the inequality holds for each pair of corresponding components.

Definition A.2.

The characteristic root λ_B of a non-negative matrix B described in Theorem A.4. is said the Frobenius-Perron root or dominant characteristic root of B.

The next theorem on M-matrixes is derived from the Frobenius-Perron's theorem.

Theorem A.5.

Let $A=(a_{ij})$ an M-matrix. Then, A has a positive characteristic root λ_A which has the following property.

(A-xii) Let ω_A a characteristic root of A and let ρ the maximum element on the main diagonal of A. Then,

$$\lambda_A \leqq \rho$$

and

$$|\rho-\omega_A| \leqq \rho-\lambda_A$$

Especially,

$$Re(\omega_A) \geqq \lambda_A$$

Definition A.3.

The characteristic root λ_A of an M-matrix A described in Theorem A.5. is said the minimum characteristic root of A.

Note 1 :

The inequalities (A-2) are not mutually independent under the assumption

$$a_{ii}>0,\ a_{ij}\leqq 0 \quad (i\neq j) \tag{A-6}$$

For example, suppose $n=3$. Then, we can induce

$$D_2>0$$

from the inequalities

$D_3 > 0$

and (A-6). (cf. footnote 3 of [34]) Therefore, in order to establish the condition (3-iii) of Theorem 3.1. or the condition (4-iii) of Theorem 4.1., it is not necessary to examine all the inequalities of (A-2), since (A-6) is guaranteed by the first two assumptions of the theorem. The problem of finding a smaller or the smallest set of inequalities, which is necessary and sufficient that a matrix $A=(a_{ij}$ satisfying (A-6) be an M-matrix, is open.

Note 2 :

The term "M-matrix" is due to Ostrowski ([42],[43], p. 259 of [22]). At first, Ostrowski defined an "M-determinant" as a determinant with non-negative off-diagonal elements whose principal minor determinants are all non-negative, and he called an M-determinant "essential" if its value is not zero, i.e. positive and "non-essential" if its value is zero. [42] Later, he used the terms "M-matrix" and "essential M-matrix" in the same meaning.[43] With those terms, he meant such a matrix,the determinant of which is an essential M-determinant in the sense defined in [42]. Here, according to Bellman [22],we define an M-matrix by Definition A.1., which is, in the result, equivalent to the definition of the M-matrix (or essential M-matrix) given by Ostrowski in [43].

Nikaido did not use the term "M-matrix", and he identified this kind of matrix with the adjective phrase "non-negatively invertible", which means the property of (A-vi).

Note 3 :

The proofs of the above theorems are found at the places given in the following. As for Theorem A.1., the equivalence of (A-0), (A-i), (A-iii), (A-v), and (A-vi) under the assumption (A-1) is proved as Theorem 6.1. and Theorem 6.2. at pp. 90-96 of [63]. The equivalence of (A-iii) and (A-ii) is mentioned in the first part of the proof of Theorem 6.1. at p. 91 of [63]. The equivalence of (A-vi) and (A-vii) under the assumption (A-1) is proved as Application 3 at p. 105 of [63]. Theorem A.2. is proved as Theorem II at p. 71 of [42]. Theorem A.4. is proved as Theorem 7.1. at p. 102 of [63]. Theorem A.5. is mentioned in Application 3 at p. 105 of [63].

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Abbreviations

ASME : The American Society of Mechanical Engineers

IECEJ : The Institute of Electronics and Communication Engineers of Japan

IEEE : The Institute of Electrical and Electronics Engineers

IEEJ : The Institute of Electrical Engineers of Japan

JAACE : The Japan Association of Automatic Control Engineers

SIAM : The Society for Industrial and Applied Mathematics